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A TREATISE

ON THE

THEORY OF DETERMINANTS

A TREATISE

ON THE

THEORY OF DETERMINANTS

WITH GRADUATED SETS OF EXERCISES

FOR USE IN COLLEGES AND SCHOOLS

BY

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London
MACMILLAN AND CO.
1882

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PREFACE.

I HAVE here attempted to do for Determinants what has been already often enough done for various other branches of Mathematics, viz. to produce a text-book containing a full exposition of the theory in a form suitable for students, and having at intervals graduated collections of exercises to test the reader's progress and to prepare him for the succeeding stage of the subject.

The First Chapter refers to determinants of the second, third, and fourth orders, but only by way of introduction to what follows, and is written in the simplest possible style. The reader who has already some acquaintance with general algebraical reasoning may pass it over.

The Second Chapter treats of determinants in general and gives in one form or another all their important properties. At first, in the demonstrations recurrence is made to the definition oftener than is necessary for the purpose of mere proof, in order that thereby the reader may become thoroughly familiarized with the definition itself, which of course really contains the whole matter.

For a similar reason the properties given towards the commencement are somewhat spread out and dwelt upon. Afterwards, the style becomes gradually more condensed to suit the advancing stage of the reader's knowledge.

The Third Chapter deals with those special forms of determinants which most commonly occur in Analysis—Continuants, Alternants, the various forms of Symmetric Determinants, Skew Determinants and Pfaffians, Compound Determinants, and the different kinds of Determinants whose elements are Differential Coefficients of a set of Functions.

The Fourth Chapter contains a summary of the History of the subject, together with a short list of previously published writings likely to be of use to the student.

I have only further to say that the book was begun to be printed in the spring of 1879, but was delayed in order that a list of all published writings on determinants might be appended. This list when completed was found to be much too lengthy to appear in such a connection, and has already been published elsewhere.

T. M.

BEECHCROFT, BISHOPTON, N.B.
24*th Dec.* 1881.

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THEORY OF DETERMINANTS.

CHAPTER I.

INTRODUCTION.

§ 1. If a particular form of algebraical expression be lengthy and of frequent occurrence, it becomes desirable to introduce a suggestive *name* and *symbol* for it; and if the form be one of a family, it is also desirable that the names and symbols of all of them should indicate this relationship. Such a nomenclature and notation are advantageous, not merely from the convenience thereby afforded in speaking and writing, but as helps to the actual discovery of the properties of the forms in question.

The following—with which the learner is probably already familiar—may be taken as an instance of this. Early in the history of the science, it being a common requirement to make use of the product resulting from the multiplication of a number by itself, this product was *named* the POWER of the number, and was *symbolized* in various ways; also the product resulting from the multiplication of the 'power' (as thus understood) of a number by the number itself, was called the CUBE of the number, and was variously expressed by means of a symbol; and similarly with other such products. Then, in the latter half of the sixteenth century, the fact that these products were members of a family was recognized in the nomenclature by calling them all POWERS, and distinguishing them as SECOND POWER, THIRD POWER, &c.; and in the next century there came into use a like improvement in notation, the second power of a, third power of a,

&c., being denoted by a^2, a^3, . . . Thence arose the subsidiary terms *exponent* and *base*, and thus the known truths regarding powers became easily expressible either in words or symbols, and the way was opened to the generalization of these truths, and to the suggestion and discovery of others.

§ 2. The expression

$$aek + dhc + gbf - gec - dbk - ahf$$

is an instance of a form which often occurs, and it is one of those with which the present text-book is especially concerned. It is seen to consist of six terms, and it involves, or is a function of, nine quantities—

$$a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ k,$$

these letters being taken in threes, in accordance with some law, to form the terms. The expression

$$x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_3y_2z_1 - x_2y_1z_3 - x_1y_3z_2$$

is another of the same kind. It is formed from the quantities

$$x_1,\ y_1,\ z_1,\ x_2,\ y_2,\ z_2,\ x_3,\ y_3,\ z_3$$

exactly as the other is formed from $a, b, c, \ldots,$ x_1 occurring in it wherever a occurs in the other, and so of $y_1, z_1, \ldots$; in other words, the second expression is the same function of $x_1, y_1, z_1, \ldots$ as the first expression is of $a, b, c, \ldots$.

§ 3. Now it has been agreed to employ for expressions of this kind a special notation. In accordance with this the first of the expressions above is denoted by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix},$$

and consequently the second by

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},$$

the nine letters being written in a certain order in three rows of three letters each, and bounded on the left by an upright line and on the right by another. Also it has been agreed that each expression when written in this form shall be called a DETERMINANT.

The comparative shortness of this notation is at once evident; the appropriateness of it and its advantages will afterwards appear. The origin of the name will also be given in its proper place.

§ 4. Comparing the form

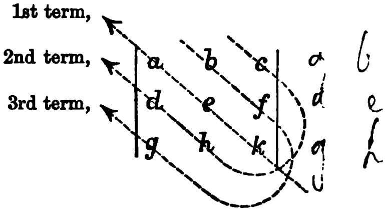
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

with the form

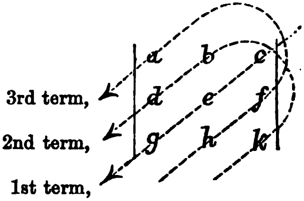
$$aek + dhc + gbf - gec - dbk - ahf,$$

we see that the first term aek is got by taking the letters which occupy the line from the left-hand top corner of the square to the right-hand bottom corner; that the letters of the second term dhc are those in a lower line parallel to this diagonal, supplemented by the letter at the right-hand top corner; that the letters of the third term gbf are those in a higher line parallel to the same diagonal, supplemented by the letter at the left-hand bottom corner; and that the next three terms, which are negative, are formed in a perfectly similar way, beginning with the other diagonal of the square.

If a line be drawn through each triad of letters forming a term, we have the following diagrams, which may assist the learner's memory, viz., for the positive terms:—



and for the negative terms—



Exercises. Set I.

Write the following algebraical expressions in the usual notation:—

1. $\begin{vmatrix} c & d & e \\ f & g & h \\ k & l & m \end{vmatrix}$

2. $\begin{vmatrix} x & y & z \\ v & w & u \\ t & r & s \end{vmatrix}$

3. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

4. $\begin{vmatrix} a & b & -c \\ d & -e & f \\ g & h & -k \end{vmatrix}$

5. $\begin{vmatrix} a & -2a & b \\ 3b & -c & 4d \\ 2c & 3d & -4b \end{vmatrix}$

6. $\begin{vmatrix} a & b & c \\ 0 & e & f \\ -g & 0 & h \end{vmatrix}$

7. $\begin{vmatrix} x & 0 & y \\ 0 & x & y \\ -x & -y & 0 \end{vmatrix}$

8. $\begin{vmatrix} x & -2z & -y^2 \\ -y & -2x & z^2 \\ -z & 2y & -x^2 \end{vmatrix}$

9. $\begin{vmatrix} -a & -b & -c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$

Find the single numbers to which the following determinants are equivalent:—

10. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

11. $\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{vmatrix}$

12. $\begin{vmatrix} 4 & 5 & 2 \\ -1 & 2 & -3 \\ 6 & -4 & 5 \end{vmatrix}$

13. $\begin{vmatrix} 1 & -1 & 1 \\ 4 & -3 & 0 \\ 3 & 2 & -5 \end{vmatrix}$ 14. $\begin{vmatrix} 4 & -1 & -2 \\ 0 & 3 & 0 \\ 3 & -7 & 4 \end{vmatrix}$ 15. $\begin{vmatrix} -1 & -1 & 1 \\ -3 & 1 & -4 \\ 2 & -3 & -5 \end{vmatrix}$

Write the following expressions in determinant form:—

16. $bfj + eid + hcg - hfd - ecj - big$.

17. $m_1n_2r_3 - m_1n_3r_2 + m_2n_3r_1 - m_2n_1r_3 + m_3n_1r_2 - m_3n_2r_1$.

Find the values of x in the following equations:—

18. $\begin{vmatrix} x & -4 & 1 \\ -6 & 3 & -2 \\ x & 2 & 1 \end{vmatrix} = 0.$ 19. $\begin{vmatrix} 1 & 1 & 1 \\ a & x & c \\ b & b & x \end{vmatrix} = 0.$

20. $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} + \begin{vmatrix} b & b & x \\ b & x & b \\ x & b & b \end{vmatrix} = 0.$

21. **Find the expression in the ordinary notation which is the equivalent of**

$$\begin{vmatrix} 2c & a+b+c & a+b+c \\ a+b+c & 2a & a+b+c \\ a+b+c & a+b+c & 2b \end{vmatrix}$$

§ 5. The expression

$$aek + dhc + gbf - gec - dbk - ahf$$

is an instance of only one of a family of forms to all of which the name *determinant* is applied, and for all of which the mode of notation which has been given above is employed. A simpler form is that in which the terms are products of *two* factors, *e.g.*,

$$ad - bc, \quad x_1y_2 - x_2y_1, \quad \ldots\ldots$$

which are denoted by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}, \quad \ldots\ldots$$

The more complicated forms are those in which the terms are products of four factors, products of five factors, and so on. The characteristics which the various forms possess in common will soon be fully referred to.

§ 6. The quantities which in the determinant notation stand unconnected in lines, and which are taken as factors to form the terms of the determinant, are called the ELEMENTS of the determinant: thus the elements of the determinant

$$\begin{vmatrix} 2a+3b & 6 & 4 \\ 0 & 1 & -3 \\ -8 & 5 & a-b \end{vmatrix}$$

are $2a+3b$, 6, 4, 0, 1, -3, -8, 5, $a-b$. The elements standing in any line from left to right constitute a ROW of the determinant; those standing in any line from top to bottom constitute a COLUMN of the determinant. The rows are numbered *first*, *second*, etc., beginning at the top, and similarly with the columns beginning on the left. Thus, in the determinant just given, the second row has the elements 0, 1, -3, and the third column has the elements 4, -3, $a-b$. In like manner the elements of a row are numbered first, second, etc., beginning on the left, and similarly with the elements of a column beginning at the top. From the appearance which a determinant presents, we are also led to speak of it as having two DIAGONALS, which are called *principal* and *secondary*, the elements standing in a line from the left-hand top corner to the right-hand bottom corner constituting the principal diagonal.

When the determinant has four, that is 2×2, elements, it is said to be of the *second* ORDER or DEGREE; when it has nine, that is 3×3, it is said to be of the *third* order or degree, and so on.

§ 7. In order to be able to derive the full benefit obtainable from the introduction of the notation of determinants, it is necessary first of all to become acquainted with their properties, and to learn how to perform with them, when possible, the various operations that fall to be performed with algebraical expressions of every kind.

§ 8. As an example of the properties of determinants, we may for the present take tho following, and establish its truth for determinants of the second and third orders.

If each element of a row of a determinant be multiplied by the same number, the determinant is thereby so multiplied.

Taking the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

and the determinant

$$\begin{vmatrix} ma & mb \\ c & d \end{vmatrix},$$

we know that the former equals

$$ad - bc,$$

and the latter

$$mad - mbc,$$

so that

$$\begin{vmatrix} ma & mb \\ c & d \end{vmatrix} = m \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Now, taking the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and the determinant

$$\begin{vmatrix} ma_1 & ma_2 & ma_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

got from the former by multiplying each element of the first row by m, we know that the first determinant equals

$$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - b_1a_2c_3 - a_1c_2b_3,$$

and the second

$$ma_1b_2c_3 + b_1c_2ma_3 + c_1ma_2b_3 - c_1b_2ma_3 - b_1ma_2c_3 - ma_1c_2b_3,$$

i.e., $m\,(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - b_1a_2c_3 - a_1c_2b_3)$,

so that

$$\begin{vmatrix} ma_1 & ma_2 & ma_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

And what has thus been shown to be true in the case of the multiplication of the elements of the first row can in the same way be shown to hold in the other cases.

Another law which we may examine in like fashion is the following:—

If a determinant be formed whose columns are in order the rows of another determinant, the two determinants will be equal.

This is at once evident for determinants of the second order.

For the case of the third order, consider the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix},$$

which we know equals

$$aek + dhc + gbf - gec - dbk - ahf.$$

When the rows are changed in order into columns, we have

$$\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & k \end{vmatrix},$$

and this in the ordinary notation is

$$aek + bfg + cdh - ceg - bdk - afh,$$

whence evidently

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & k \end{vmatrix}.$$

EXERCISES. SET II.

1. Find what relation exists between the two algebraical expressions

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}, \quad \begin{vmatrix} a & b & c \\ g & h & k \\ d & e & f \end{vmatrix}.$$

2. Find how the expressions

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}, \quad \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & k \end{vmatrix}, \quad \begin{vmatrix} d & e & f \\ g & h & k \\ a & b & c \end{vmatrix}$$

are related to each other.

3. State the probable theorem regarding determinants, to which the results of Ex. 1 and 2 point.

4. Show that

$$\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} = 0.$$

5. Without changing from the determinant notation, show that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} = 0,$$

by using what has already been proved.

6. Enunciate the probable theorem of which the identities of Ex. 4 and 5 are particular cases.

7. Show that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1 \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - y_1 \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} + z_1 \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix},$$

$$= x_1 \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - x_2 \begin{vmatrix} y_1 & z_1 \\ y_3 & z_3 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}.$$

8. Supply the elements in the following blank determinant-forms of the second order:—

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = z_1 \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} - z_2 \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} + z_3 \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix},$$

$$= -x_2 \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} + y_2 \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix} - z_2 \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix}.$$

9. Show that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ x_1 & y_1 \end{vmatrix} \begin{vmatrix} a_2 & b_2 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 \\ x_2 & y_2 \end{vmatrix} \begin{vmatrix} x_1 & y_1 \\ a_2 & b_2 \end{vmatrix},$$

$$= \begin{vmatrix} x_1 & b_1 \\ x_2 & b_2 \end{vmatrix} \begin{vmatrix} a_1 & y_1 \\ a_2 & y_2 \end{vmatrix} + \begin{vmatrix} y_1 & b_1 \\ y_2 & b_2 \end{vmatrix} \begin{vmatrix} x_1 & a_1 \\ x_2 & a_2 \end{vmatrix}.$$

10. Show that

$$\begin{vmatrix} a + x & b & c \\ d + y & e & f \\ g + z & h & k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} + \begin{vmatrix} x & b & c \\ y & e & f \\ z & h & k \end{vmatrix}.$$

11. Show that

$$\begin{vmatrix} a + mc & b & c \\ d + mf & e & f \\ g + mk & h & k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} a - mb & b & c \\ d - me & e & f \\ g - mh & h & k \end{vmatrix}.$$

12. Without changing the first determinant into the ordinary notation, show that

$$\begin{vmatrix} a\mathrm{A} + b\mathrm{B} & a\mathrm{C} + b\mathrm{D} \\ c\mathrm{A} + d\mathrm{B} & c\mathrm{C} + d\mathrm{D} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{vmatrix}.$$

13. Without passing from the determinant notation, show that

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0.$$

Write the following expressions in determinant form :—

14. $ayp - myc + xnc - anz + mbz - bxp.$

15. $x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3.$

16. $3abc - a^3 - b^3 - c^3.$

17. $acf + 2bed - cd^2 - b^2f - ae^2.$

Find single determinants of the third order equivalent to the expressions—

18. $$x\begin{vmatrix} x & c \\ c & x \end{vmatrix} - a\begin{vmatrix} a & b \\ c & x \end{vmatrix} + b\begin{vmatrix} a & b \\ x & c \end{vmatrix}.$$

19. $$\begin{vmatrix} 0 & a_2 & a_4 \\ b_1 & 0 & b_4 \\ c_1 & c_2 & 0 \end{vmatrix} + a_1\begin{vmatrix} 0 & b_4 \\ c_2 & 0 \end{vmatrix} + b_2\begin{vmatrix} 0 & a_4 \\ c_1 & 0 \end{vmatrix} + c_4\begin{vmatrix} 0 & a_2 \\ b_1 & 0 \end{vmatrix} + a_1b_2c_4.$$

20. $$\left\{\begin{vmatrix} a & b \\ d & e \end{vmatrix}\begin{vmatrix} e & f \\ y & z \end{vmatrix} - \begin{vmatrix} d & e \\ x & y \end{vmatrix}\begin{vmatrix} b & c \\ e & f \end{vmatrix}\right\} \div e.$$

§ 9. As an instance of how determinants come into use in Algebra, there may be taken the case of the solution of a set of simultaneous equations of the first degree.

If

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \; \ldots . \; (1) \\ \text{and} \qquad a_2x + b_2y &= c_2 \; \ldots . \; (2) \end{aligned}\right\}$$

then we have

$$\left.\begin{aligned} a_1b_2x + b_1b_2y &= b_2c_1 \\ -a_2b_1x - b_1b_2y &= -b_1c_2 \end{aligned}\right\}$$

Hence, by addition,

$$(a_1b_2 - a_2b_1)\,x = b_2c_1 - b_1c_2,$$

$$\therefore \qquad x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}},$$

and, similarly,
$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Again, if

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \quad \ldots\ldots (1) \\ a_2x + b_2y + c_2z &= d_2 \quad \ldots\ldots (2) \\ a_3x + b_3y + c_3z &= d_3 \quad \ldots\ldots (3) \end{aligned}\right\}$$

then, using

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \quad -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

as multipliers, we have

$$\left.\begin{aligned} a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}x + b_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}y + c_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}z &= d_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \\ -a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}x - b_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}y - c_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}z &= -d_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \\ a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}x + b_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}y + c_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}z &= d_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \end{aligned}\right\}$$

Hence, by addition, we have (Exercises, Set II. 7)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}x + \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}y + \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}z = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$

Now, the second and third determinants in this equation are each equal to 0 (Exercises, Set II. 6);

$$\therefore \qquad x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

In a similar way, or, more shortly, by using the result just obtained, we may show that

$$y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and

$$z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The learner should compare the values of x, y, z just found with those of x, y in the case of the preceding set of equations, noting that the denominator is always the determinant whose elements are in order the coefficients of the unknown quantities in the given equations, and that the numerator of the value of any of the unknown quantities differs from the denominator simply in having the right-hand members of the equations occupying in order the places of the coefficients of the unknown quantity in question.

§ 10. One advantage of these solutions lies in the fact that the results obtained are such as can be exceedingly easily remembered, so that we are thus enabled to derive the benefit usually attached to remembered results, viz., of being able to utilize them in the solution of similar problems. Thus, if the given set of equations be

$$\left.\begin{aligned} 3x + 2y - 4z &= -5 \\ 2x - 3y + z &= -1 \\ 4x + y - 2z &= 0 \end{aligned}\right\}$$

and it be required to find the value of y, we have at once

$$\begin{aligned} y &= \begin{vmatrix} 3 & -5 & -4 \\ 2 & -1 & 1 \\ 4 & 0 & -2 \end{vmatrix} \div \begin{vmatrix} 3 & 2 & -4 \\ 2 & -3 & 1 \\ 4 & 1 & -2 \end{vmatrix}, \\ &= \frac{6 + 0 - 20 - 16 - 20 - 0}{18 - 8 + 8 - 48 + 8 - 3}, \\ &= \frac{-50}{-25}, \\ &= 2. \end{aligned}$$

EXERCISES. SET III.

Tell immediately the values of x and y which satisfy the following pairs of equations:—

1. $\left.\begin{aligned} 4x + 3y &= 24 \\ 5x + 2y &= 23 \end{aligned}\right\}$

2. $\left.\begin{aligned} 3x + 5y &= 17 \\ 2x + 3y &= 11 \end{aligned}\right\}$

3. $\left.\begin{aligned} 6x - 4y &= 6 \\ 7x - 3y &= 12 \end{aligned}\right\}$

4. $\left.\begin{aligned} 4x - 5y &= 15 \\ -3x + 17y &= 2 \end{aligned}\right\}$

5. $\left.\begin{aligned} -ax + by &= a^2 \\ bx - ay &= -b^2 \end{aligned}\right\}$

6. $\left.\begin{aligned} -4x + 7y - 10 &= 0 \\ 7x - 4y + 1 &= 0 \end{aligned}\right\}$

Find, by means of determinants, the values of x, y, and z which satisfy the following sets of equations:—

7. $\left.\begin{aligned} 3x - 4y + 2z &= 1 \\ 2x + 3y - 3z &= -1 \\ 5x - 5y + 4z &= 7 \end{aligned}\right\}$

8. $\left.\begin{aligned} 3x + 4y - 5z &= -2 \\ 4x + 5y - 3z &= 11 \\ 5x + 3y - 4z &= 3 \end{aligned}\right\}$

9. $\left.\begin{aligned} 4x - 7y + z &= 16 \\ 3x + y - 2z &= 10 \\ 5x - 6y - 3x &= 10 \end{aligned}\right\}$

10. $\left.\begin{aligned} 6x + 8y + 3z &= 6 \\ 5x + 6y - 9z &= 1 \\ 7x - 10y - 3z &= 0 \end{aligned}\right\}$

11.
$$\left.\begin{array}{r} 5x - 4z = 42 \\ 3z + 5y = 1 \\ 4y - 3x = -10 \end{array}\right\}$$

12.
$$\left.\begin{array}{l} \dfrac{6}{x} - \dfrac{2}{y} + \dfrac{1}{z} = 4 \\ \dfrac{2}{x} + \dfrac{5}{y} - \dfrac{2}{z} = \frac{3}{4} \\ \dfrac{5}{x} - \dfrac{1}{y} + \dfrac{3}{z} = 6\frac{3}{4} \end{array}\right\}$$

13. Having given

$$\left.\begin{array}{l} a_1x + b_1y + c_1 = 0 \;.\;.\;.\;.\; (1) \\ a_2x + b_2y + c_2 = 0 \;.\;.\;.\;.\; (2) \\ a_3x + b_3y + c_3 = 0 \;.\;.\;.\;.\; (3) \end{array}\right\}$$

show, by solving for x and y in (2) and (3) and substituting the results in (1), that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

§ 11. Knowing that by definition

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

and that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = aek + dhc + gbf - gec - dbk - ahf,$$

we are now naturally led to inquire what the *general* definition of a determinant is, and what, in accordance with this definition, is the expression of the fourth degree denoted by

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix}.$$

It is evident that the definition must inform us on two points, viz., first, how the *terms* are got from the elements of the determinant; and, second, how the *signs* of the terms are fixed.

§ 12. The law of the composition of the terms of a determinant is that in every term there shall be present as a factor one, and only one, element from each row of the determinant, these being so chosen that there shall also be only one from each column; and every product composed of elements in accordance with this law is a term of the determinant. Thus, in the case of the above determinant of the fourth order,

$$bipg$$

must be a term, for there is in it from the first column i, from the second b, from the third g, from the fourth p; and from the first row b, from the second g, from the third i, from the fourth p; that is, one and only one from each row and column. Also, we have clearly an example in the elements of the principal diagonal, which constitute a term known as the *principal diagonal term.*

§ 13. Having got one term of a given determinant, it is easy, as will now be seen, to find others by means of it, and in this way ultimately to arrive at them all. Let us consider the determinant of the fourth degree

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix}$$

and begin with the diagonal term

$$afkp. \qquad (1)$$

Taking af to be part of another term, and seeking to find other two elements to complete it, we at once see that we cannot get them from the first or second row, nor from the first or second column; that is to say, our choice is restricted as shown by the deletions in the diagram

$$\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} & \cancel{d} \\ \cancel{e} & \cancel{f} & \cancel{g} & \cancel{h} \\ \cancel{i} & \cancel{j} & k & l \\ \cancel{m} & \cancel{n} & o & p \end{vmatrix}.$$

Now, having already taken kp along with af, it is clear that there only remains ol to be taken; thus we have the term

$$afol, \tag{2}$$

and know likewise that there can be no other term with the elements a, f occurring in it. Proceeding in the same way, we now take ao to be part of another term which we wish to complete, and, looking back to the determinant, we see that a being in the first row and first column, and o in the fourth row and third column, these rows and columns must be left out of our choice. Deleting as before, we therefore have

$$\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} & \cancel{d} \\ \cancel{e} & f & \cancel{g} & h \\ \cancel{i} & j & \cancel{k} & l \\ \cancel{m} & \cancel{n} & \cancel{o} & \cancel{p} \end{vmatrix},$$

whence it is evident that, besides the term $afol$, only one other can contain the elements a, o, viz.,

$$ajoh. \tag{3}$$

Again, retaining a, j, two of the elements of the term last found, our determinant, with deletions, is

$$\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} & \cancel{d} \\ \cancel{e} & \cancel{f} & g & h \\ \cancel{i} & \cancel{j} & \cancel{k} & \cancel{l} \\ \cancel{m} & \cancel{n} & o & p \end{vmatrix},$$

and, as before, it is clear that, besides the term $ajoh$, there is only one other containing the elements a, j, viz., the term

$$ajgp. \qquad (4)$$

Similarly, retaining the elements a, g of this term, we obtain another term,

$$angl, \qquad (5)$$

and, retaining the part an of this, we arrive at the other term containing this part, viz.,

$$ankh. \qquad (6)$$

If now, in continuing our process, we still select a as one of the elements to be retained, we shall find that no new term is to be got from any of the above six, and that indeed no other term of the determinant contains a. But, retaining two elements other than a, say n, k, we find the new term

$$enkd,$$

and proceeding as before, we readily obtain all the other terms containing e. Then we should pass to the terms containing i, and from these to the terms containing m; and this would complete the work, for by the definition every term must contain one of the four a, e, i, m.

EXERCISES. SET IV.

In the above determinant of the fourth degree, *mbkh* is a term; find—

1. the other terms involving m;
2. the other terms involving b;
3. the other terms involving k;
4. the other terms involving h.

5. From the term *mbkh* derive any term involving n, and thence all the other terms involving n.

6. Similarly find all the terms involving o.

7. Similarly find all the terms involving p.

In the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 & v_1 & w_1 \\ x_2 & y_2 & z_2 & v_2 & w_2 \\ x_3 & y_3 & z_3 & v_3 & w_3 \\ x_4 & y_4 & z_4 & v_4 & w_4 \\ x_5 & y_5 & z_5 & v_5 & w_5 \end{vmatrix}$$

find—

8. the terms containing $x_5 w_1$;
9. the terms containing $y_2 w_3$;
10. the terms containing $x_4 v_1$.

§ 14. We come now to consider the fixing of the *signs* of the terms. Taking as an example the case of the term

$$bipg,$$

mentioned in § 12, we proceed as follows. Finding the numbers of the rows from which the elements b, i, p, g are taken, and also the numbers of the columns, we note them down in separate lines in the order in which the elements occur, the result being

$$\left.\begin{matrix} 1, & 3, & 4, & 2, \\ 2, & 1, & 4, & 3. \end{matrix}\right\}$$

Now, looking at the first of these lines, and contrasting the order in which the numbers come with the natural order

1, 2, 3, 4, we observe that 3 precedes 2 instead of following it, and that there is only one other such *inversion* (as it is called) in the line, viz., that in which the 4 and 2 are concerned. Noting this, we proceed to the other set of numbers, and find that the number of inversions of order in it is also two. The total number of inversions in the two sets is therefore *four*, and it is this number which fixes whether the term $bipg$ is positive or negative, the sign-factor being $(-1)^4$. Had the number of inversions of order been *five*, the sign-factor would have been $(-1)^5$, and so on; in other words, the sign is + or − according as the number of inversions of order is even or odd.

The process necessary for fixing the sign of a term is thus seen to consist in writing down in order the numbers specifying the rows from which the elements in the term are taken, and in another line the numbers specifying the columns, and counting how many instances there are in each line of a number preceding another which it would follow if the numbers were in their natural order, the sign-factor of the term being $(-1)^n$ if n be the total number of such instances.

This process may be shortened if we first arrange the elements of the term so that the element which comes from the first row is placed first, the element from the second row placed second, and so on; because then the number of inversions of order in the first of the two series will be 0, and we shall have only to count the number of inversions in the second series. Thus, writing the term above considered in the form

$$bgip,$$

the numbers indicating the rows and columns are

$$\left.\begin{matrix} 1, & 2, & 3, & 4, \\ 2, & 3, & 1, & 4, \end{matrix}\right\}$$

so that the number of inversions of order is

$$0 + 2,$$

and we have the sign + as before.

§ 15. If we now take the terms of the above determinant of the fourth degree in the order in which they are found in § 13, we see at once that $afkp$ is positive—indeed it is evident that the sign of the principal diagonal term must in every case be positive—that the next term $afol$ is negative, the next $ajoh$ positive, the next $ajgp$ negative, and so on. It would thus appear that, beginning with the principal diagonal term, we may with ease deduce from it all the other terms, each of them with its proper sign. The principal diagonal term is on this account also called the *leading term* of the determinant.

EXAMPLE.—Find the number of inversions of order in the series

$$7, 8, 4, 1, 3, 2, 9, 6, 5.$$

Taking 7 along with each of the numbers which follow it, we have the couplets

$$(7, 8),\ (7, 4),\ (7, 1),\ (7, 3),\ (7, 2),\ (7, 9),\ (7, 6),\ (7, 5),$$

and of these it is evident that *six* are instances of inversion of order; taking 8 along with each of the numbers which follow it, we obtain other *six* inversions; and proceeding in like manner with 4 and the other numbers of the series, we find that the total number of inversions

$$\begin{aligned} &= 6 + 6 + 3 + 0 + 1 + 0 + 2 + 1, \\ &= 19. \end{aligned}$$

EXERCISES. SET V.

1. Count the number of inversions of order in

$$\begin{gathered} 3,\ 6,\ 4,\ 1,\ 5,\ 2; \\ 7,\ 1,\ 6,\ 5,\ 3,\ 4,\ 2; \\ 3,\ 2,\ 9,\ 4,\ 1,\ 6,\ 7; \\ 4,\ 8,\ 6,\ 7,\ 2,\ 5,\ 3; \\ 7,\ 8,\ 3,\ 2,\ 1,\ 4,\ 6,\ 5; \\ 3,\ 1,\ 9,\ 8,\ 2,\ 6,\ 5,\ 7,\ 4. \end{gathered}$$

2. Tell the signs of the terms

$$ingd,\ lgbm,\ gdjm,\ nkah$$

of the determinant

$$\begin{vmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{vmatrix}.$$

3. Find the full expression for the preceding determinant in the ordinary notation.

4. Tell the signs of the terms

$$a_5b_4c_3d_2e_1,\quad a_3e_2c_4d_1b_5,\quad b_2a_1e_3c_4d_5,\quad b_3c_2a_1d_5e_4$$

of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}.$$

5. Show that the determinant of Ex. 2 and 3 above

$$= a\begin{vmatrix} f & j & n \\ g & k & o \\ h & l & p \end{vmatrix} - e\begin{vmatrix} b & j & n \\ c & k & o \\ d & l & p \end{vmatrix} + i\begin{vmatrix} b & f & n \\ c & g & o \\ d & h & p \end{vmatrix} - m\begin{vmatrix} b & f & j \\ c & g & k \\ d & h & l \end{vmatrix},$$

$$= a\begin{vmatrix} f & j & n \\ g & k & o \\ h & l & p \end{vmatrix} - b\begin{vmatrix} e & i & m \\ g & k & o \\ h & l & p \end{vmatrix} + c\begin{vmatrix} e & i & m \\ f & j & n \\ h & l & p \end{vmatrix} - d\begin{vmatrix} e & i & m \\ f & j & n \\ g & k & o \end{vmatrix}.$$

6. Show from the preceding that

$$\begin{vmatrix} xa & e & i & m \\ xb & f & j & n \\ xc & g & k & o \\ xd & h & l & p \end{vmatrix} = x\begin{vmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{vmatrix}.$$

7. Prove that

$$\begin{vmatrix} a+x & e & i & m \\ b+y & f & j & n \\ c+z & g & k & o \\ d+w & h & l & p \end{vmatrix} = \begin{vmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{vmatrix} + \begin{vmatrix} x & e & i & m \\ y & f & j & n \\ z & g & k & o \\ w & h & l & p \end{vmatrix}.$$

8. **Prove that**

$$\begin{vmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{vmatrix} = \begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix}.$$

9. **Prove that**

$$\begin{vmatrix} a & e & i & i \\ b & f & j & j \\ c & g & k & k \\ d & h & l & l \end{vmatrix} = 0.$$

10. **Prove that**

$$\begin{vmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{vmatrix} = -\begin{vmatrix} a & e & m & i \\ b & f & n & j \\ c & g & o & k \\ g & h & p & l \end{vmatrix}.$$

CHAPTER II.

DETERMINANTS IN GENERAL.

§ 16. *A* DETERMINANT *consists of such a number of quantities so situated that they may be viewed as arranged either in successive lines (called* rows) *running from left to right and each containing as many quantities as there are lines, or in successive lines (called* columns) *running perpendicular to the former; and it is used to denote the expression which consists of all the terms that can be formed by taking the product of as many quantities as there are rows—one quantity from each row and thereby one from each column, the sign preceding any term being determined by writing in succession the numbers of the rows from which the quantities composing it have come, and in a separate series the numbers of the columns, and taking* + *or* – *according as the total number of inversions of order in these two series is even or odd.*

The terms referred to are spoken of as the terms of the determinant, and the expression which consists of all the terms is called the *expansion* or *ordinary expansion* of the determinant.

The rest of the subsidiary nomenclature has been already fully explained in § 6.

§ 17. *If all the elements of a row or column of a determinant be zero, so also is the determinant itself.*

§ 18. *The full number of terms of a determinant of the* n^{th} *order is* $1.2.3.\ldots n$.

If in forming any term we be careful to take the elements which are to compose it from the rows in order, viz., the first element of the term from the first row, the second element from the second row, and so on, then the first element may be taken from any of the n columns, the second element from any of the n columns except the column from which the first element has been taken, the third element from any of the n columns except the columns which have already been drawn upon, and so on. Hence the numbers of the columns from which the elements necessary to form a term are taken constitute a permutation of the numbers

$$1,\ 2,\ 3,\ 4,\ \ldots\ldots,\ n,$$

and every such permutation gives rise to a term. Therefore the number of terms is the same as the number of permutations of the first n integers, i.e., $1.2.3.\ldots n$.

§ 19. When the elements of a determinant are suffixed letters, only one letter being used in each row, and the set of suffix numbers for all the rows being the same and in order of magnitude, the finding of the expansion of the determinant is more readily accomplished, both as to the formation of the terms and as to the fixing of the signs, than in cases where suffixed letters are not employed. For in the symbol denoting an element the letter will indicate the row to which the element belongs, and the suffix will indicate the column, so that to form a term we have only to write all the letters in succession and attach one of the suffixes to each, and to fix the sign of the term we have only to count the number of inversions of order in the suffixes as there written. Thus, in the case of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

we write all the permutations of the suffixes, viz.,

$$\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \\ 2 & 1 & 3, \end{array}$$

and taking each permutation along with the letters a, b, c, we have at once the expansion

$$a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_2 + a_3b_2c_1 - a_2b_3c_1 - a_2b_1c_3.$$

§ 20. It is also possible in the case of such determinants to use an abridged notation. One form of this consists in writing one of the terms, viz., the leading term, as a type of them all, prefixing to it the symbol ± to indicate the variability of the signs, enclosing this in brackets, and before all placing the symbol of summation Σ. Thus the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

is denoted by

$$\Sigma\,(\pm\, a_1b_2c_3).$$

Equally efficient notations are

$$D(a_1 b_2 c_3), \quad |a_1 b_2 c_3|,$$

which do not, like the other, aim at a partial definition, but are meant merely as suggestive contractions for the longer form, or for *the determinant of which the first row consists of* a's, *the second row of* b's, *the third row of* c's, *and of which the elements in the first column have the suffix* 1, *those in the second column the suffix* 2, *and those in the third column the suffix* 3. A fourth but less distinctive form than these is also in use when no ambiguity is likely to result, viz., for the case of the same determinant

$$(a_1 b_2 c_3).$$

The learner should accustom himself to the use of these shorter forms, and especially to pass mentally from them with ease to the standard notation. Thus, to take another instance,

$$D(x_0 y_2 z_4 w_6), \quad \Sigma(\pm x_0 y_2 z_4 w_6), \quad \text{or} \quad |x_0 y_2 z_4 w_6|$$

should readily suggest

$$\begin{vmatrix} x_0 & x_2 & x_4 & x_6 \\ y_0 & y_2 & y_4 & y_6 \\ z_0 & z_2 & z_4 & z_6 \\ w_0 & w_2 & w_4 & w_6 \end{vmatrix}.$$

§ 21. LEMMA ON INVERSIONS OF ORDER.—*If in a series of integers which are all different any adjacent pair be transposed, the number of inversions of order is thereby increased or diminished by one.*

Let the series of integers be

$$m,\ p,\ \ldots\ r,\ \beta,\ \phi,\ t,\ \ldots\ n,\ s,$$

and let β and ϕ be transposed, so that we have

$$m,\ p,\ \ldots\ r,\ \phi,\ \beta,\ t,\ \ldots\ n,\ s.$$

In counting the number of inversions in the first series, we take m along with each of the integers following m, and note how many of the couplets thus got are instances of inversion of order, then we take p along with the integers following p, and so on. Now it is clear that the couplets beginning with m are the same for both series, and that indeed no difference can occur until we have finished the couplets beginning with r. Similarly it is clear that the couplets beginning with t are the same for both series, and that so also are the couplets beginning with any of the integers after t. Any difference that may exist is thus shown to be confined to the couplets beginning with β and with ϕ. Examining these we finally see that the couplets for both series are exactly the same, except that for (β, ϕ) in the first we have (ϕ, β) in the second. Now if (β, ϕ) be an inversion of order, (ϕ, β) is not, and *vice versâ;* hence the first series has either one inversion more or one less than the second.

§ 22. *In a determinant of the* n^th^ *degree two and not more than two terms can have* $n-2$ *elements in common, and of such a pair of terms the one is positive and the other negative.*

Suppose that in the process of forming a term we have taken $n-2$ elements out of $n-2$ rows and $n-2$ columns, and that the rows which have not been drawn upon are the

b^{th} and p^{th}, and the columns the d^{th} and t^{th}. To complete the term we have only four elements to choose from, viz.,

$$\left.\begin{array}{rl} & \text{the element of the } b^{\text{th}} \text{ row and } d^{\text{th}} \text{ column,}\\ & \text{the element of the } b^{\text{th}} \text{ row and } t^{\text{th}} \text{ column,}\\ & \text{the element of the } p^{\text{th}} \text{ row and } d^{\text{th}} \text{ column,}\\ \text{and} & \text{the element of the } p^{\text{th}} \text{ row and } t^{\text{th}} \text{ column.}\end{array}\right\}$$

It is therefore clear that we can have one complete term by taking

$$\left.\begin{array}{rl} & \text{the element of the } b^{\text{th}} \text{ row and } d^{\text{th}} \text{ column,}\\ \text{and} & \text{the element of the } p^{\text{th}} \text{ row and } t^{\text{th}} \text{ column;}\end{array}\right\}$$

and another by taking

$$\left.\begin{array}{rl} & \text{the element of the } b^{\text{th}} \text{ row and } t^{\text{th}} \text{ column,}\\ \text{and} & \text{the element of the } p^{\text{th}} \text{ row and } d^{\text{th}} \text{ column;}\end{array}\right\}$$

and that no other selection is possible.

The sign of the former term is fixed by the number of inversions of order in the series

$$\left.\begin{array}{l} \ldots\ldots\ b,\ p,\\ \ldots\ldots\ d,\ t;\end{array}\right\}$$

the sign of the latter is similarly dependent on the series

$$\left.\begin{array}{l} \ldots\ldots\ b,\ p,\\ \ldots\ldots\ t,\ d;\end{array}\right\}$$

the portions indicated by the dots being the same for both terms. This shows that there is one more inversion in the one case than in the other, and consequently the sign of the one term is + and of the other −.

§ 23. *Of the full number of terms of a determinant, exactly as many are positive as are negative.*

Suppose the positive terms of the determinant are taken and placed in a column, the elements in each term being arranged in the order of the rows from which they come, and that a line is drawn cutting off the last two elements in each term, thus:—

$$\begin{array}{r|l} bef\ldots\ldots ak & lm \\ cgh\ldots\ldots dn & pq \\ \cdots\cdots\cdots\cdots & \cdots \\ \cdots\cdots\cdots\cdots & \cdots \end{array}$$

Then we know that the portions to the left of the line are all different, because if two terms were alike in this portion, the one would be positive and the other negative (§ 22), which is not the case. Further, corresponding to each term in the column a negative term may be found differing from the positive term in the last two elements (§ 22), and all these negative terms would be different, for, as we have seen, they would all be different even if we looked only at the elements preceding the last two in each. There are thus at least as many negative terms as positive. In the same way we can show that there are as many positive terms as negative; and thus the theorem is established.

EXERCISES. SET VI.

Find directly from the definition the expansion of the determinants—

1. $\begin{vmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & b_1 & a_2 \\ 0 & c_1 & b_2 & a_3 \\ d_1 & c_2 & b_3 & a_4 \end{vmatrix}$.

2. $\begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & 0 & 0 & d_2 \\ 0 & 0 & 0 & d_3 \end{vmatrix}$.

3. $\begin{vmatrix} b_2 & b_3 & b_4 & b_5 \\ 0 & c_3 & 0 & 0 \\ 0 & d_3 & d_4 & d_5 \\ 0 & e_3 & 0 & e_5 \end{vmatrix}$.

4. $$\begin{vmatrix} x & 0 & 0 & 0 & y \\ y & x & 0 & 0 & 0 \\ 0 & y & x & 0 & 0 \\ 0 & 0 & y & x & 0 \\ 0 & 0 & 0 & y & x \end{vmatrix}.$$

5. $$\begin{vmatrix} a_4 & 0 & c_2 & 0 & x \\ a_3 & 0 & c_1 & x & d_1 \\ a_2 & 0 & x & 0 & 0 \\ a_1 & x & b_1 & b_2 & b_3 \\ x & 0 & 0 & 0 & 0 \end{vmatrix}.$$

6. Write the terms of $|x_0y_1z_2w_3|$ which contain y_2w_3.

7. What other term of $|b_0c_1d_2e_3f_4|$, besides the secondary diagonal term, contains the elements $f_0c_3e_1$?

8. Find the number of inversions of order in

$$n,\ n-1,\ n-2,\ \ldots\ldots,\ 3,\ 2,\ 1.$$

9. What other term of $|a_mb_nc_pd_qe_r|$ besides $a_nb_pc_md_re_q$ contains the elements $b_pc_me_q$?

10. What is the sign-factor of the secondary diagonal term in a determinant of the n^{th} order?

11. Write the terms of $|a_1b_3c_5d_6|$ which contain the element b_5.

12. If the integral numbers

$$d,\ e,\ f,\ g,\ h,\ i,\ j,\ k$$

be 'cyclically' transposed so as to become

$$k,\ d,\ e,\ f,\ g,\ h,\ i,\ j,$$

how many more or fewer inversions of order will there be?

13. Find how many terms of a determinant of the n^{th} order contain any particular element.

14. If in a series of integers which are all different *any* pair be transposed, the number of inversions of order is thereby increased or diminished by an odd number.

15. If the sign of a term be determined from one arrangement of the elements composing it, show that the same sign would be got from a different arrangement.

§ 24. *Two determinants, which differ only in that the rows of the one are in order the columns of the other, are equal.*

Every term of the first determinant must contain one and only one element from each row and each column of that determinant; therefore it must contain one and only one element from each column and each row of the second determinant, and therefore it must be a term of that determinant also. Similarly we can show that every term of the second determinant is a term of the first; therefore the terms of the two determinants are alike in magnitude.

Now suppose that the first element in any term of the first determinant is from the a^{th} row and α^{th} column, the second element from the b^{th} row and β^{th} column, and so on; then the sign of the term is fixed by the number of inversions in the two series

$$\left.\begin{array}{l} a,\ b,\ c,\ \ldots, \\ \alpha,\ \beta,\ \gamma,\ \ldots. \end{array}\right\}$$

But the element which is in the a^{th} row and α^{th} column in the first determinant is in the α^{th} row and a^{th} column in the second determinant, and so of the other elements. Hence the sign of the term in question in the second determinant is fixed by the number of inversions in the two series

$$\left.\begin{array}{l} \alpha,\ \beta,\ \gamma,\ \ldots, \\ a,\ b,\ c,\ \ldots. \end{array}\right\}$$

And the number of inversions here being the same as before, the sign of the term is the same in both determinants.

Thus the two determinants being shown to be alike both in magnitude and sign, the theorem is established.

§ 25. From this it is evident that any theorem, in the statement of which the word *row* or the words *row* and *column* occur would also be true if the word *column* or the words *column* and *row* respectively were substituted. For in proving the former theorem in regard to the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ b_1 & b_2 & b_3 & \ldots & b_n \\ c_1 & c_2 & c_3 & \ldots & c_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & l_3 & \ldots & l_n \end{vmatrix}$$

we are proving the latter theorem in regard to the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & \dots & l_1 \\ a_2 & b_2 & c_2 & \dots & l_2 \\ a_3 & b_3 & c_3 & \dots & l_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & l_n \end{vmatrix}.$$

§ 26. *If all the elements of a row of a determinant be multiplied by the same quantity, the resulting determinant equals the product of the original determinant and the said quantity.*

Let m be the multiplier referred to. Then, if we form any term of the original determinant, and form at the same time the corresponding term of the new determinant, no difference can possibly occur unless at the row mentioned, where the element taken from the new determinant is m multiplied by the element taken from the original determinant. Now since one element and only one must be taken from this row or column for the formation of every term of the determinants, it follows that each term of the new determinant is m multiplied by the corresponding term of the other. Hence the new determinant equals the original determinant multiplied by m.

EXAMPLES :—

$$m\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} ma & mb & mc \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} a & mb & c \\ d & me & f \\ g & mh & k \end{vmatrix},$$

$$= \frac{1}{m}\begin{vmatrix} a & mb & c \\ md & m^2e & mf \\ g & mh & k \end{vmatrix} = m\begin{vmatrix} a & mb & c \\ \frac{d}{m} & e & \frac{f}{m} \\ g & mh & k \end{vmatrix} = \dots .$$

$$\begin{vmatrix} 1 & 7 & -1 \\ 3 & 21 & 0 \\ 4 & -28 & -6 \end{vmatrix} = 7 \begin{vmatrix} 1 & 1 & -1 \\ 3 & 3 & 0 \\ 4 & -4 & -6 \end{vmatrix} = 7 \times 2 \begin{vmatrix} 1 & 1 & -1 \\ 3 & 3 & 0 \\ 2 & -2 & -3 \end{vmatrix},$$

$$= 7 \times 2 \times 3 \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & -2 & -3 \end{vmatrix},$$

$$= 7 \times 2 \times 3 \times (-3 + 2 + 2 + 3),$$

$$= 168.$$

§ 27. *If two rows of a determinant be identical, the determinant is equal to zero.*

Let the determinant be

$$\begin{vmatrix} \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_1 \quad a_2 \ldots a_h \ldots a_r \ldots a_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_1 \quad a_2 \ldots a_h \ldots a_r \ldots a_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{vmatrix},$$

the two rows which are identical being the b^{th} and p^{th}; and, thinking of any term whatever of the determinant, suppose that the element taken from the b^{th} row to form it is a_h and the element from the p^{th} row a_r, so that the term may be represented by

$$Aa_h a_r,$$

A standing for the product of all the elements of the term except a_h and a_r. Then proceeding as in § 22 to find the other term which contains A we obtain

$$Aa_r a_h,$$

which equals the former term, but must (§ 22) be of opposite sign. Thus the positive and negative terms of the determinant are equal in magnitude: hence the truth of the theorem.

§ 28. *If each of the elements of a row of a determinant consist of two terms, the determinant may be expressed as the sum of two determinants, the first of which is got from the original determinant by excluding one term of each of the elements in question, and the second by replacing these and excluding the other terms.*

If $A+B$ be one of the binomial elements referred to, then it is clear that if we fix on any term containing $A+B$ in the original determinant, we shall find the corresponding term of one of the pair of determinants to differ from this term only in having A for $A+B$, and the corresponding term of the other determinant to differ from it only in having B for $A+B$. That is to say, any term of the original determinant being

$$(A+B)\,efk\ldots,$$

the corresponding terms of the pair of determinants are

$$Aefk\ldots \text{ and } Befk\ldots$$

Hence the first determinant equals the sum of the two other determinants.

EXAMPLES:—

$$\begin{vmatrix} a & \mathrm{A+B} & h \\ b & \mathrm{C+D} & k \\ c & \mathrm{E-F} & l \end{vmatrix} = \begin{vmatrix} a & A & h \\ b & C & k \\ c & E & l \end{vmatrix} + \begin{vmatrix} a & B & h \\ b & D & k \\ c & -F & l \end{vmatrix},$$

and

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g+h & k & l+m \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & k & l \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ h & 0 & m \end{vmatrix}.$$

§ 29. More generally, if each of the elements of a row consist of n terms, it is evident that the given determinant can be partitioned in similar fashion into n determinants.

Further, if the elements of one row be n-termed and the elements of another row be m-termed, the determinant can be partitioned into n determinants, each of which will have a row consisting of m-termed elements and each of which can therefore be partitioned into m determinants, so that finally the original determinant can be expressed as the sum of $m \times n$ determinants.

More generally still, if the elements of the first row consist each of m_1 terms, the elements of the second row of m_2 terms, and so on, the determinant can be partitioned into $m_1 m_2 m_3 \ldots$ determinants.

EXAMPLE:—

$$\begin{vmatrix} a_1+b_1+c_1 & d_1 & e_1-f_1 \\ a_2+b_2+c_2 & d_2 & e_2-f_2 \\ a_3+b_3+c_3 & d_3 & e_3-f_3 \end{vmatrix} = \begin{vmatrix} a_1 & d_1 & e_1-f_1 \\ a_2 & d_2 & e_2-f_2 \\ a_3 & d_3 & e_3-f_3 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 & e_1-f_1 \\ b_2 & d_2 & e_2-f_2 \\ b_3 & d_3 & e_3-f_3 \end{vmatrix} + \begin{vmatrix} c_1 & d_1 & e_1-f_1 \\ c_2 & d_2 & e_2-f_2 \\ c_3 & d_3 & e_3-f_3 \end{vmatrix},$$

$$= \left\{ \begin{array}{l} \begin{vmatrix} a_1 & d_1 & e_1 \\ a_2 & d_2 & e_2 \\ a_3 & d_3 & e_3 \end{vmatrix} - \begin{vmatrix} a_1 & d_1 & f_1 \\ a_2 & d_2 & f_2 \\ a_3 & d_3 & f_3 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 & e_1 \\ b_2 & d_2 & e_2 \\ b_3 & d_3 & e_3 \end{vmatrix} \\ - \begin{vmatrix} b_1 & d_1 & f_1 \\ b_2 & d_2 & f_2 \\ b_3 & d_3 & f_3 \end{vmatrix} + \begin{vmatrix} c_1 & d_1 & e_1 \\ c_2 & d_2 & e_2 \\ c_3 & d_3 & e_3 \end{vmatrix} - \begin{vmatrix} c_1 & d_1 & f_1 \\ c_2 & d_2 & f_2 \\ c_3 & d_3 & f_3 \end{vmatrix}. \end{array} \right\}$$

§ 30. The identity established in § 28 may be otherwise viewed as a theorem for the *addition* of two or more determinants which are related to each other in a particular way; that is to say, beginning with the second member of the identity, the theorem is:—*The sum of any number of determinants which are alike except as regards a particular row, the* r^th^ *say, is equal to a determinant which is like each of the given determinants except that any element of its* r^th^ *row is the sum of the corresponding elements of all the given determinants.*

EXAMPLES :—

$$\begin{vmatrix} 4 & 3 & 4 \\ 7 & -1 & 3 \\ 2 & 8 & 6 \end{vmatrix} + \begin{vmatrix} 4 & 3 & 4 \\ 5 & 1 & -3 \\ 2 & 8 & 6 \end{vmatrix} + \begin{vmatrix} 5 & -12 & 3 \\ 3 & 0 & 8 \\ 4 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 3 & 4 \\ 12 & 0 & 0 \\ 2 & 8 & 6 \end{vmatrix} + \begin{vmatrix} 5 & 3 & 4 \\ -12 & 0 & 0 \\ 3 & 8 & 6 \end{vmatrix},$$

$$= \begin{vmatrix} 9 & 3 & 4 \\ 0 & 0 & 0 \\ 5 & 8 & 6 \end{vmatrix},$$

$$= 0.$$

$$m|a_1b_2c_3d_4| + n|a_1b_2c_3e_4| = |a_1b_2c_3\overline{md_4+ne_4}|.$$

§ 31. *If the first element of a row of a determinant be increased by any multiple of the first element of any other row, the second element of the former row be increased by the same multiple of the second element of the latter, and so on with all the other elements of the two rows, the new determinant thus obtained is equal to the original one.*

Let the given determinant be

$$\left|\begin{array}{cccc} \cdot & \cdot & \cdot & \cdots \\ a & b & c & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ x & y & z & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{array}\right| \quad \text{or} \quad \Delta,$$

and let the common multiplier be m, so that the new determinant is

$$\left|\begin{array}{cccc} \cdot & \cdot & \cdot & \cdots \\ a+mx & b+my & c+mz & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ x & y & z & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{array}\right| \quad \text{or} \quad \Delta'.$$

On account of the row of binomial elements in Δ' we have (§ 28)—

$$\Delta' = \begin{vmatrix} \cdot & \cdot & \cdot & \cdots \\ a & b & c & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ x & y & z & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} + \begin{vmatrix} \cdot & \cdot & \cdot & \cdots \\ mx & my & mz & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ x & y & z & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix},$$

$$= \begin{vmatrix} \cdot & \cdot & \cdot & \cdots \\ a & b & c & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ x & y & z & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} + m \begin{vmatrix} \cdot & \cdot & \cdot & \cdots \\ x & y & z & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ x & y & z & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix}, \qquad (\S\ 26),$$

$$= \Delta + m \times 0, \qquad (\S\ 27),$$

$$= \Delta.$$

§ 32. If another row of the determinants Δ, Δ' in the preceding be

$$p,\ q,\ r,\ \ldots\ .$$

it follows that

$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdots \\ a+mx+np & b+my+nq & c+mz+nr & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ x & y & z & \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} = \Delta',$$

and therefore also $= \Delta$;

so that from a continued application of the theorem of § 31 we have a more general result, viz.: *If the elements of any row of a determinant be increased by any equimultiples of the corresponding elements of a second row and by any equimultiples of the corresponding elements of a third row, and so on, the resulting determinant is equal to the original one.*

§ 33. This theorem may be advantageously employed in the simplification of determinants, more especially of those whose elements are expressed in figures.

For example, consider the determinant

$$\begin{vmatrix} 14 & 15 & 11 \\ 21 & 22 & 16 \\ 23 & 29 & 17 \end{vmatrix}.$$

Subtracting each element of the third column from the corresponding element of the first and second columns we have the equivalent determinant

$$\begin{vmatrix} 3 & 4 & 11 \\ 5 & 6 & 16 \\ 6 & 12 & 17 \end{vmatrix},$$

and this we know (§ 26) is equal to

$$2\begin{vmatrix} 3 & 2 & 11 \\ 5 & 3 & 16 \\ 6 & 6 & 17 \end{vmatrix}.$$

Now subtracting each element of the second column from the corresponding element of the first column, and multiplying each element of the second column by 5 and subtracting the result from the corresponding element of the third column we have

$$2\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 0 & 6 & -13 \end{vmatrix},$$

which if we please we may alter similarly into

$$2\begin{vmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 6 & -13 \end{vmatrix},$$

the given determinant being thus equal to

$$2(13 + 6) \quad \textit{i.e.,} \quad 38.$$

Operating on the rows instead of the columns, we might have proceeded thus:—

$$\begin{vmatrix} 14 & 15 & 11 \\ 21 & 22 & 16 \\ 23 & 29 & 17 \end{vmatrix} = \begin{vmatrix} 14 & 15 & 11 \\ 7 & 7 & 5 \\ 2 & 7 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 4 \\ 2 & 7 & 1 \end{vmatrix},$$

$$= 35 + 8 - 5 \quad = 38,$$

as before.

Example 1. Show that

$$\begin{vmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{vmatrix} = (a+b+c+d)(-a+b-c+d)\begin{vmatrix} 0 & 1 & -1 & 1 \\ 1 & c & d & a \\ 1 & d & a & b \\ 1 & a & b & c \end{vmatrix}.$$

$$\text{The given determinant} = \begin{vmatrix} a+b+c+d & b & c & d \\ b+c+d+a & c & d & a \\ c+d+a+b & d & a & b \\ d+a+b+c & a & b & c \end{vmatrix}, \qquad (\S\ 32),$$

$$= \begin{vmatrix} 1 & b & c & d \\ 1 & c & d & a \\ 1 & d & a & b \\ 1 & a & b & c \end{vmatrix} (a+b+c+d), \qquad (\S\ 26),$$

$$= \begin{vmatrix} 0 & b-c+d-a & c-d+a-b & d-a+b-c \\ 1 & c & d & a \\ 1 & d & a & b \\ 1 & a & b & c \end{vmatrix} (a+b+c+d),$$

$$= \begin{vmatrix} 0 & 1 & -1 & 1 \\ 1 & c & d & a \\ 1 & d & a & b \\ 1 & a & b & c \end{vmatrix} (a+b+c+d)\,(-a+b-c+d).$$

EXAMPLE 2. Show that

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = (b-a)\,(c-a)\,(c-b)\,(d-a)\,(d-b)\,(d-c),$$

without finding the expansion of the determinant.

If each element of the first row be subtracted from the corresponding element of the second row, it is at once clear that $b-a$ is a factor of the determinant, and that $c-a$, $c-b$, $d-a$, $d-b$, $d-c$ are factors follows equally readily in the same way. Further we see that, if these factors were multiplied together, the first term of the product would be bc^2d^3, and that all the other terms would be unlike this, so that the coefficient of bc^2d^3 in the product is $+1$. On the other hand, looking at the principal diagonal, we see that the coefficient of bc^2d^3 in the expansion of the determinant is also $+1$. Thus the identity is established.

EXAMPLE 3. Show that

$$\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & \gamma^2 & \beta^2 \\ b^2 & \gamma^2 & 0 & \alpha^2 \\ c^2 & \beta^2 & \alpha^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & a\alpha & b\beta & c\gamma \\ a\alpha & 0 & c\gamma & b\beta \\ b\beta & c\gamma & 0 & a\alpha \\ c\gamma & b\beta & a\alpha & 0 \end{vmatrix}.$$

Taking the first determinant and multiplying the elements of its first and second rows by α, and the elements of its third and fourth rows by a, we obtain as its equivalent

$$\frac{1}{a^2\alpha^2} \begin{vmatrix} 0 & a^2\alpha & b^2\alpha & c^2\alpha \\ a^2\alpha & 0 & \gamma^2\alpha & \beta^2\alpha \\ b^2a & \gamma^2a & 0 & \alpha^2a \\ c^2a & \beta^2a & \alpha^2a & 0 \end{vmatrix},$$

and dividing the elements of the first and second *columns* of this new determinant by a, and the elements of the third and fourth by α, we thus have the original determinant equal to

$$\begin{vmatrix} 0 & a\alpha & b^2 & c^2 \\ a\alpha & 0 & \gamma^2 & \beta^2 \\ b^2 & \gamma^2 & 0 & a\alpha \\ c^2 & \beta^2 & a\alpha & 0 \end{vmatrix}.$$

Continuing in exactly similar fashion we find it

$$= \frac{1}{b^2\beta^2}\begin{vmatrix} 0 & a\alpha\beta & b^2\beta & c^2\beta \\ a\alpha b & 0 & \gamma^2 b & \beta^2 b \\ b^2\beta & \gamma^2\beta & 0 & a\alpha\beta \\ c^2 b & \beta^2 b & a\alpha b & 0 \end{vmatrix} = \begin{vmatrix} 0 & a\alpha & b\beta & c^2 \\ a\alpha & 0 & \gamma^2 & b\beta \\ b\beta & \gamma^2 & 0 & a\alpha \\ c^2 & b\beta & a\alpha & 0 \end{vmatrix},$$

$$= \frac{1}{c^2\gamma^2}\begin{vmatrix} 0 & a\alpha\gamma & b\beta\gamma & c^2\gamma \\ a\alpha c & 0 & \gamma^2 c & b\beta c \\ b\beta c & \gamma^2 c & 0 & a\alpha c \\ c^2\gamma & b\beta\gamma & a\alpha\gamma & 0 \end{vmatrix} = \begin{vmatrix} 0 & a\alpha & b\beta & c\gamma \\ a\alpha & 0 & c\gamma & b\beta \\ b\beta & c\gamma & 0 & a\alpha \\ c\gamma & b\beta & a\alpha & 0 \end{vmatrix},$$

as was required.

By a further combination of multiplications and divisions the process assumes a neater form, thus:—Taking for the rows the multipliers

$$\alpha\beta\gamma,\quad abc,\quad \alpha\beta c,\quad ab\gamma,$$

respectively, that is, multiplying in all by $(abc\alpha\beta\gamma)^2$, we have

$$\begin{vmatrix} 0 & a^2\alpha\beta\gamma & b^2\alpha\beta\gamma & c^2\alpha\beta\gamma \\ a^2abc & 0 & \gamma^2abc & \beta^2abc \\ b^2\alpha\beta c & \gamma^2\alpha\beta c & 0 & a^2\alpha\beta c \\ c^2ab\gamma & \beta^2ab\gamma & a^2ab\gamma & 0 \end{vmatrix},$$

and then all that is required is to divide by $(abc\alpha\beta\gamma)^2$ by operating on the *columns* with the divisors

$$abc,\quad \alpha\beta\gamma,\quad ab\gamma,\quad \alpha\beta c,$$

respectively.

EXERCISES. SET VII.

Find the simplest forms of the following numerical expressions:—

1. $\begin{vmatrix} 15 & 17 & 16 \\ 12 & 18 & 14 \\ 19 & 17 & 13 \end{vmatrix}.$ 2. $\begin{vmatrix} 15 & 13 & 10 \\ 12 & 17 & 10 \\ 16 & 11 & 19 \end{vmatrix}.$ 3. $\begin{vmatrix} 20 & 15 & 25 \\ 17 & 12 & 22 \\ 19 & 20 & 16 \end{vmatrix}.$

4. $\begin{vmatrix} 27 & 37 & 47 \\ 33 & 23 & 29 \\ 25 & 28 & 24 \end{vmatrix}$. 5. $\begin{vmatrix} 22 & 29 & 27 \\ 25 & 23 & 30 \\ 28 & 26 & 24 \end{vmatrix}$. 6. $\begin{vmatrix} 30 & 36 & 35 \\ 33 & 31 & 37 \\ 38 & 34 & 32 \end{vmatrix}$.

7. What effect is produced on a determinant of the n^{th} degree by multiplying all its elements by -1?

8. Find the simplified expansion of the determinant

$$\begin{vmatrix} a & a+3 & a+6 \\ a+1 & a+4 & a+7 \\ a+2 & a+5 & a+8 \end{vmatrix}.$$

9. Show that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1+z \end{vmatrix} = xyz.$$

10. Find the simplified expansion of

$$\begin{vmatrix} a_1-b_1 & a_2-b_2 & a_3-b_3 \\ b_1-c_1 & b_2-c_2 & b_3-c_3 \\ c_1-a_1 & c_2-a_2 & c_3-a_3 \end{vmatrix}, \quad \text{and of} \quad \begin{vmatrix} a+c & 2a-b & b+2c \\ b+a & 2b-c & c+2a \\ c+b & 2c-a & a+2b \end{vmatrix}.$$

11. Prove that, if the sum or difference of every pair of corresponding elements of two rows of a determinant be a constant multiple of the corresponding element of another row, the determinant is equal to zero.

12. Express

$$\begin{vmatrix} a_1+h_1+k_1 & a_2+h_1+k_2 & a_3+h_1+k_3 & 1 \\ b_1+h_2+k_1 & b_2+h_2+k_2 & b_3+h_2+k_3 & 1 \\ c_1+h_3+k_1 & c_2+h_3+k_2 & c_3+h_3+k_3 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

in a simpler form as a determinant of the fourth order.

13. Show that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \frac{1}{a_1a_2a_3}\begin{vmatrix} 1 & 1 & 1 \\ a_2a_3b_1 & a_1a_3b_2 & a_1a_2b_3 \\ a_2a_3c_1 & a_1a_3c_2 & a_1a_2c_3 \end{vmatrix}.$$

14. Show that

$$\begin{vmatrix} 0 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & a_3b_2c_1 & a_2b_3c_1 \\ 1 & a_3b_1c_2 & a_2b_1c_3 \end{vmatrix}.$$

15. Solve the equation

$$\begin{vmatrix} a_1+b_1x & c_1 & d_1 \\ a_2+b_2x & c_2 & d_2 \\ a_3+b_3x & c_3 & d_3 \end{vmatrix} = 0.$$

16. Prove that

$$\begin{vmatrix} a+c & b+d & a+c & b+d \\ b+d & a+c & b+d & a+c \\ a+b & b+c & c+d & d+a \\ c+d & d+a & a+b & b+c \end{vmatrix} = 0.$$

17. Find the simplified expansion of

$$\begin{vmatrix} x & a & b & c+d \\ x & b & c & d+a \\ x & c & d & a+b \\ x & d & a & b+c \end{vmatrix}, \quad \text{and of} \quad \begin{vmatrix} a & b+c+d & a+b & c+d \\ b & c+d+a & b+c & d+a \\ c & d+a+b & c+d & a+b \\ d & a+b+c & d+a & b+c \end{vmatrix}.$$

18. Express

$$\begin{vmatrix} a_2x^2+a_1x+a_0 & a_3 & a_4 \\ b_2x^2+b_1x+b_0 & b_3 & b_4 \\ c_2x^2+c_1x+c_0 & c_3 & c_4 \end{vmatrix}$$

in terms arranged according to ascending powers of x.

19. Prove that, if the sum or difference of every pair of corresponding elements of two rows of a determinant be a constant multiple of the sum or difference of the corresponding pair of elements of two other rows, the determinant is equal to zero.

20. Show that

$$\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}.$$

21. Express either determinant of the preceding exercise as the product of four linear factors.

22. Prove that a determinant remains substantially the same if the signs of the elements be changed in every alternate member of the set of lines consisting of either diagonal and the lines parallel to it, the diagonal itself being one of the lines left unaltered.

23. Prove that

$$\begin{vmatrix} a & b & c & 0 \\ b & a & 0 & c \\ c & 0 & a & b \\ 0 & c & b & a \end{vmatrix} = \begin{vmatrix} -a & b & c & 0 \\ b & -a & 0 & c \\ c & 0 & -a & b \\ 0 & c & b & -a \end{vmatrix}.$$

Resolve the following determinants into simple factors:—

$$24.\ \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}. \qquad 25.\ \begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}.$$

$$26.\ \begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} \qquad 27.\ \begin{vmatrix} x^3 & ax^2 & a^2x & a^3 \\ y^3 & by^2 & b^2y & b^3 \\ z^3 & cz^2 & c^2z & c^3 \\ w^3 & dw^2 & d^2w & d^3 \end{vmatrix}.$$

28. Show that

$$\begin{vmatrix} bcd & a & a^2 & a^3 \\ cda & b & b^2 & b^3 \\ dab & c & c^2 & c^3 \\ abc & d & d^2 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 & a^4 \\ 1 & b^2 & b^3 & b^4 \\ 1 & c^2 & c^3 & c^4 \\ 1 & d^2 & d^3 & d^4 \end{vmatrix}.$$

Find a single determinant equivalent to

29. $| a_1b_4c_5 | + | a_2b_4c_5 | - | a_3b_4c_5 |$.

30. $| a_0b_2c_5 | - | a_0b_3c_5 | - | a_1b_3c_5 | + | a_1b_2c_5 |$.

§ 34. *If two adjacent rows of a determinant be transposed, the new determinant differs only in sign from the original one.*

Suppose that the rows which change places are the m^{th} and $(m+1)^{\text{th}}$, and that the elements taken from these rows towards the formation of a particular term $\ldots eh\beta\tau ps\ldots$ of the original determinant are β and τ, viz., β from the m^{th} row and τ from the $(m+1)^{\text{th}}$. Then $\ldots eh\beta\tau ps\ldots$ is a term of the new determinant as well, for there is in it one and only one element from each row of that determinant, viz., β from the $(m+1)^{\text{th}}$, τ from the m^{th}, and from the other rows exactly the same elements as in the case of the original determinant. Further, $\ldots eh\beta\tau ps\ldots$ contains one and only one element from each column of the new determinant, for the columns in order furnish each the same element as did those of the original determinant, the transposition of two rows not affecting the number of the column to which an element may belong. It follows therefore that $\ldots eh\beta\tau ps\ldots$ is a term of the new determinant.

For fixing its sign as a term of this determinant we know that the series of numbers indicating the columns from which the elements come is exactly the same as in the case of the original determinant, and that the series indicating the rows differs only in that two consecutive numbers have changed places. Consequently the difference in the number of inversions in the two cases is 1 (§ 21), and therefore the sign must be different in the one case from what it is in the other.

Every term of the original determinant being thus shown to occur with a different sign in the new determinant, and the number of terms being the same in both cases, it follows that the two determinants differ only in sign.

EXAMPLES :—

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = -\begin{vmatrix} a & b & c \\ g & h & k \\ d & e & f \end{vmatrix} = \begin{vmatrix} b & a & c \\ h & g & k \\ e & d & f \end{vmatrix} = \ldots\ldots,$$

and

$$| a_1b_2c_3d_4 | = - | a_1b_3c_2d_4 | = | a_1b_3c_4d_2 | = - | a_1c_3b_4d_2 | = \ldots\ldots$$

§ 35. *If from a determinant* Δ *another determinant* Δ' *be got as if by making one of the rows of the former pass from its place over* p *rows, then* $\Delta = (-1)^p \Delta'$.

The transference may be effected by transposition of the row in question with the p rows in succession, beginning with the nearest of them. This would occasion p changes of sign; hence the truth of the theorem.

EXAMPLES:—

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ k & l & m & n \\ o & p & q & r \end{vmatrix} = (-1)^3 \begin{vmatrix} b & c & d & a \\ f & g & h & e \\ l & m & n & k \\ p & q & r & o \end{vmatrix} = (-1)^5 \begin{vmatrix} b & c & d & a \\ p & q & r & o \\ f & g & h & e \\ l & m & n & k \end{vmatrix} = \ldots.;$$

$$| a_1b_2c_3d_4e_5 | = (-1)^3 | a_1b_5c_2d_3e_4 | = (-1)^7 | a_4b_1c_5d_2e_3 | = \ldots.;$$

and

$$\begin{aligned} | a_1b_2c_3d_4e_5 | + | a_1e_2b_3c_4d_5 | &= | a_1b_2c_3d_4e_5 | + (-1)^3 | a_1b_2c_3d_4e_5 |, \\ &= 0. \end{aligned}$$

§ 36. *If any two rows of a determinant be transposed, the new determinant differs only in sign from the original one.*

Suppose that r rows lie between the two rows, A and B say, referred to; and denote the original determinant by Δ and the other by Δ'. Δ' may be got from Δ by making the row A pass over the r rows and thus come alongside of the row B, and then making the row B pass over the row A and the r rows. By the first operation r changes of sign are occasioned and by the second $r+1$ changes; that is, in all $2r+1$. Hence

$$\begin{aligned} \Delta' &= (-1)^{2r+1}\Delta, \\ &= -\Delta, \end{aligned}$$

since the index $2r+1$ is an odd number.

EXAMPLE:—

$$| a_1b_2c_3d_4e_5 | = - | a_1e_2c_3d_4b_5 | = | a_4e_2c_3d_1b_5 | = \ldots.$$

§ 37. The theorem (§ 27) in regard to the result of the identity of two rows of a determinant is usually established by means of the foregoing theorem. The mode of proof, which is worth the learner's attention, is as follows:—

Let the determinant be Δ. Then, transposing the two rows referred to, we get a determinant which is equal to $-\Delta$ (§ 36). But this new determinant is exactly the same as the original, on account of the identity of the two rows transposed. Hence we have

$$\Delta = -\Delta,$$

$$\text{so that} \quad 2\Delta = 0,$$

$$\text{and} \quad \therefore \quad \Delta = 0.$$

§ 38. *If two determinants* Δ, Δ' *of the* n*th* *degree be such that the first row of the one is the same as the last row of the other, the second row of the one the same as the* (n−1)*th* *row of the other, the third row of the one the same as the* (n−2)*th* *row of the other, and so on, then* $\Delta = (-1)^{\frac{1}{2}n(n-1)}\Delta'$.

The transformation of the one determinant into the other may be effected by making the first row of Δ pass over the $n-1$ other rows, then making what was before this the second row pass over $n-2$ rows, next making what was formerly the third row pass over $n-3$ rows, and so on until what was originally the $(n-1)$th row is made to pass over the one next it, viz. that which originally was the nth. The number of changes of sign consequent upon these alterations is

$$(n-1)+(n-2)+(n-3)+\ldots.+2+1.$$

But the sum of this equidifferent progression is

$$\tfrac{1}{2}n(n-1);$$

hence the truth of the theorem.

EXAMPLES:—

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = (-1)^{\frac{4}{2}(4-1)} \begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ c_1 & c_2 & c_3 & c_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \end{vmatrix} = \begin{vmatrix} d_1 & d_2 & d_3 & d_4 \\ c_1 & c_2 & c_3 & c_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \end{vmatrix};$$

and

$$| a_1b_2c_3d_4e_5 | = (-1)^{\frac{5}{2}(5-1)} | e_1d_2c_3b_4a_5 | = | e_1d_2c_3b_4a_5 | ;$$

$$| a_1b_2c_3d_4e_5f_6 | = (-1)^{\frac{6}{2}(6-1)} | f_1e_2d_3c_4b_5a_6 | = - | f_1e_2d_3c_4b_5a_6 | .$$

§ 39. *If two determinants* Δ, Δ' *of the* n^{th} *degree be such that the first row of the one is when reversed the last row of the other, the second row of the one when reversed the* $(n-1)^{th}$ *row of the other, the third row of the one when reversed the* $(n-2)^{th}$ *row of the other, and so on, then* $\Delta = \Delta'$.

The transformation of the one determinant into the other may be effected by reversing the order of the rows and then in the result reversing the order of the columns. The number of changes of sign occasioned by this is (§ 38)

$$\tfrac{1}{2}n(n-1) + \tfrac{1}{2}n(n-1),$$

i.e.
$$n(n-1);$$

$$\therefore \quad \Delta = (-1)^{n(n-1)}\Delta'.$$

Now one of the consecutive integers $n, n-1$ must be even, and consequently their product must be even: hence

$$\Delta = \Delta'.$$

EXAMPLES:—

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = - \begin{vmatrix} g & h & k \\ d & e & f \\ a & b & c \end{vmatrix} = \begin{vmatrix} k & h & g \\ f & e & d \\ c & b & a \end{vmatrix}.$$

and
$$| a_1b_2c_3d_4 | = | d_4c_3b_2a_1 | .$$

§ 40. *A determinant being given, it is possible to transfer any element to the place occupied by any other, and yet have the resulting determinant equal in magnitude to the original one.*

If the two elements be in the same row it is at once seen that the transposition of the columns to which they belong effects the change referred to; and, contrariwise, if the two elements be in the same column.

If the two elements be neither in the same row nor in the same column, what is necessary is the transposition of the rows they belong to, followed, in the form which results, by the transposition of the columns.

For example, if we wish the element b_2 of the determinant $|a_1b_2c_3d_4e_5|$ to be in the fourth row and fifth column, we proceed as follows :—

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} = \begin{vmatrix} a_1 & a_5 & a_3 & a_4 & a_2 \\ d_1 & d_5 & d_3 & d_4 & d_2 \\ c_1 & c_5 & c_3 & c_4 & c_2 \\ b_1 & b_5 & b_3 & b_4 & b_2 \\ e_1 & e_5 & e_3 & e_4 & e_2 \end{vmatrix}.$$

The attainment of the end desired is dependent upon the transference of one row to the place of another, and, as this may be accomplished in other ways than by the transposition of the two rows, there is a corresponding possible variety in the form of the results. Thus

$$| a_1b_2c_3d_4e_5 | = | a_1c_2d_3b_4e_5 | = - | a_1c_3d_4b_5e_2 |,$$

or, transposing cyclically,

$$| a_1b_2c_3d_4e_5 | = | d_1e_2a_3b_4c_5 | = | d_3e_4a_5b_1c_2 |.$$

EXERCISES. SET VIII.

1. Without finding the expansions of the determinants show that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} h & g & k \\ e & d & f \\ b & a & c \end{vmatrix}.$$

2. Show that

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = \begin{vmatrix} a & c & b & d \\ i & k & j & l \\ e & g & f & h \\ m & o & n & p \end{vmatrix}.$$

3. Show that

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \begin{vmatrix} c_1 & b_1 & a_1 & d_1 \\ c_4 & b_4 & a_4 & d_4 \\ c_3 & b_3 & a_3 & d_3 \\ c_2 & b_2 & a_2 & d_2 \end{vmatrix}.$$

4. Show that

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ 0 & j & 0 & 0 \\ k & l & m & n \end{vmatrix} = \begin{vmatrix} j & 0 & 0 & 0 \\ f & e & g & h \\ b & a & c & d \\ l & k & m & n \end{vmatrix}.$$

5. Show that

$$| a_1 b_2 c_3 d_4 | = | d_1 b_4 c_3 a_2 |,$$

and

$$| a_1 b_2 c_3 d_4 e_5 | = | a_3 b_2 e_1 d_4 c_5 |.$$

Find by cyclical transposition of the rows and columns of $|b_0 c_1 d_2 e_3 f_4|$ a determinant equal to the said determinant and having

6. d_3 in the first row and first column;
7. e_1 in the first row and first column;
8. c_2 in the fifth row and second column;
9. f_4 in the second row and third column.

10. Show that

$$\begin{vmatrix} 0 & b_1 & b_2 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ d_1 & d_2 & d_3 & d_4 \\ 0 & c_1 & c_2 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & a_1 & a_4 & a_3 \\ b_1 & 0 & 0 & b_2 \\ c_1 & 0 & 0 & c_2 \\ d_2 & d_1 & d_4 & d_3 \end{vmatrix}.$$

11. Show that

$$\begin{vmatrix} l & 0 & 0 & 0 \\ d & a & c & b \\ k & 0 & h & 0 \\ g & 0 & f & e \end{vmatrix} = \begin{vmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & k \\ 0 & 0 & 0 & l \end{vmatrix}.$$

12. Show that

$$\begin{vmatrix} c_3 & c_6 & c_4 & c_2 & c_5 \\ 0 & a_6 & 0 & a_2 & 0 \\ d_3 & d_6 & d_4 & d_2 & d_5 \\ 0 & b_6 & 0 & b_2 & 0 \\ e_3 & e_6 & e_4 & e_2 & e_5 \end{vmatrix} = \begin{vmatrix} a_6 & a_2 & 0 & 0 & 0 \\ b_6 & b_2 & 0 & 0 & 0 \\ c_6 & c_2 & c_3 & c_4 & c_5 \\ d_6 & d_2 & d_3 & d_4 & d_5 \\ e_6 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}.$$

13. Transform

$$\begin{vmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{vmatrix}$$

so as to have the principal diagonal composed (1) of the four a's, (2) of the four b's, (3) of the four c's, (4) of the four d's.

14. Show that

$$\begin{vmatrix} a_1+a_2 & a_2+a_3 & a_3+a_1 \\ b_1+b_2 & b_2+b_3 & b_3+b_1 \\ c_1+c_2 & c_2+c_3 & c_3+c_1 \end{vmatrix} = 2\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

15. Show that

$$\begin{vmatrix} a_1+a_2+a_3 & a_2+a_3+a_4 & a_3+a_4+a_1 & a_4+a_1+a_2 \\ b_1+b_2+b_3 & b_2+b_3+b_4 & b_3+b_4+b_1 & b_4+b_1+b_2 \\ c_1+c_2+c_3 & c_2+c_3+c_4 & c_3+c_4+c_1 & c_4+c_1+c_2 \\ d_1+d_2+d_3 & d_2+d_3+d_4 & d_3+d_4+d_1 & d_4+d_1+d_2 \end{vmatrix} = 3\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

16. Use the principle employed in § 37 to show that

$$\begin{vmatrix} 0 & a & b & c & d \\ -a & 0 & e & f & g \\ -b & -e & 0 & h & i \\ -c & -f & -h & 0 & j \\ -d & -g & -i & -j & 0 \end{vmatrix} = 0.$$

17. Similarly show that

$$\begin{vmatrix} a & b & c & d & 0 \\ e & f & g & 0 & d \\ h & i & 0 & g & -c \\ j & 0 & i & -f & b \\ 0 & j & -h & e & -a \end{vmatrix} = 0.$$

18. If m rows, viz., the h_1th, h_2th, . . . , h_mth, be transferred so as to become the 1st, 2nd, . . . , mth, without altering the relative positions of the remaining rows, and then n columns, viz., the k_1th, k_2th, . . . , k_nth, be similarly transferred, the determinant thus obtained is the same as the original or differs from it only in sign according as

$$h_1+h_2+\ldots+h_m-\tfrac{1}{2}m(m+1)+k_1+k_2+\ldots+k_n-\tfrac{1}{2}n(n+1)$$

is even or odd.

19. Without finding the expansion of the determinant show that $ab+bc+ca$ is a factor of

$$\begin{vmatrix} ab & c^2 & c^2 \\ a^2 & bc & a^2 \\ b^2 & b^2 & ac \end{vmatrix}.$$

Establish the following identities :—

20. $$\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 4abc.$$

21. $$\begin{vmatrix} \dfrac{a^2+b^2}{c} & c & c \\ a & \dfrac{b^2+c^2}{a} & a \\ b & b & \dfrac{c^2+a^2}{b} \end{vmatrix} = 4abc.$$

22. $$\begin{vmatrix} a+b-c & c & c \\ a & b+c-a & a \\ b & b & c+a-b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

23. $$\begin{vmatrix} (a+b)^2 & ca & bc \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc\,(a+b+c)^3.$$

24. $$\begin{vmatrix} a+b+nc & (n-1)a & (n-1)b \\ (n-1)c & b+c+na & (n-1)b \\ (n-1)c & (n-1)a & c+a+nb \end{vmatrix} = n(a+b+c)^3.$$

25. $$\begin{vmatrix} (a+b)^3 & -c^3 & -c^3 \\ -a^3 & (b+c)^3 & -a^3 \\ -b^3 & -b^3 & (c+a)^3 \end{vmatrix} = 3abc(a+b+c)^3\Sigma a^2b.$$

§ 41. *If all the elements of the first row of a determinant be zero except the first and it be* 1, *the determinant equals the determinant of lower degree got by deleting the first row and column.*

Let

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \dots \\ b & e & f & g & h \dots \\ c & k & \dots & \dots & \dots \\ d & l & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} e & f & g & h \dots \\ k & \dots & \dots & \dots \\ l & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

be the determinants referred to, and denote them by Δ and Δ' respectively.

No terms can exist in Δ containing any element of the first column except the first element. For any term containing any other element, c say, of the first column must, like every other term, contain an element of the first row; and that element cannot be the element 1, for 1 belongs to the same column as c; therefore it must be 0, and thus the term vanishes. Consequently to form a term of the determinant Δ we must begin with the element 1, and for the other elements must take one from every row and column except the first row and column; that is to say, one from every row and column of the determinant Δ'. We thus see that the other elements of the term constitute a term of Δ'; and the element already mentioned being 1, it follows that every term of Δ is a term of Δ'.

But further, every term of Δ' is a term of Δ. For any term of Δ' contains one and only one element from every row and column of Δ except the first row and column; so that to make it a term of Δ we only need to annex an element from the first row and column of Δ; and this element being 1, the term is not altered.

Not only, however, have the two determinants the same terms, but the terms which are equal in magnitude are alike in sign. Suppose the two series which fix the sign of any term in Δ' to be

$$\left.\begin{matrix} a, & \kappa, & \tau, & \lambda, & \ldots \\ \delta, & \beta, & \sigma, & \gamma, & \ldots \end{matrix}\right\},$$

then to find the series which fix the sign of the equal term in Δ, we must prefix the numbers indicating the row and column to which the additional element belongs. But these numbers being both 1, the number of inversions of order remains unchanged.

§ 42. Reversing the order of the members of the identity which has just been established, we may view the theorem as affirming that *without altering the value of a determinant its order may be raised by superposing a zero on every column and prefixing a* 1 *to the row of zeros thus formed and an element of any finite magnitude whatever to each of the other rows.* Thus

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ \alpha & a & b & c \\ \beta & d & e & f \\ \gamma & g & h & k \end{vmatrix} = \begin{vmatrix} 1 & \mathrm{A} & \mathrm{B} & \mathrm{C} \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & k \end{vmatrix},$$

$$= \begin{vmatrix} 1 & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathrm{G} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \alpha & a & b & c \\ 0 & \beta & d & e & f \\ 0 & \gamma & g & h & k \end{vmatrix} = \ldots\ldots$$

§ 43. *If any element of a determinant* Δ *be* 1, *say the element which is in the* p^{th} *row and* q^{th} *column, and the other elements of the same row be* 0, *then, denoting by* Δ' *the determinant of lower degree which is got by deleting the* p^{th} *row and* q^{th} *column,* $\Delta = (-1)^{\mathrm{p}+\mathrm{q}}\Delta'$.

Passing the p^{th} row over the $p-1$ rows which precede it thereby making it the first row, and passing the q^{th} column of the result over the $q-1$ columns which precede it thereby making it the first column, we do not alter the relative position of any of the elements outside the original p^{th} row and q^{th} column. There is thus obtained a determinant of the form in § 41, differing from Δ by the multiplier $(-1)^{p-1+q-1}$, and such that when the first row and column of it are omitted the resulting determinant is still the same determinant as would have been got had the p^{th} row and q^{th} column been deleted before the transformation. Hence

$$\begin{aligned} \Delta &= (-1)^{p+q-2}\Delta', \\ &= (-1)^{p+q}\Delta'. \end{aligned}$$

EXAMPLES:—

$$\begin{vmatrix} a & b & c \\ 0 & 1 & 0 \\ d & e & f \end{vmatrix} = -\begin{vmatrix} 0 & 1 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = (-1)^2\begin{vmatrix} 1 & 0 & 0 \\ b & a & c \\ e & d & f \end{vmatrix} = \begin{vmatrix} a & c \\ d & f \end{vmatrix}.$$

$$\begin{vmatrix} a & b & 0 & d \\ c & e & 0 & f \\ g & h & 1 & k \\ l & m & 0 & n \end{vmatrix} = (-1)^2\begin{vmatrix} 0 & a & b & d \\ 0 & c & e & f \\ 1 & g & h & k \\ 0 & l & m & n \end{vmatrix} = (-1)^4\begin{vmatrix} 1 & g & h & k \\ 0 & a & b & d \\ 0 & c & e & f \\ 0 & l & m & n \end{vmatrix}$$

$$= \begin{vmatrix} a & b & d \\ c & e & f \\ l & m & n \end{vmatrix}.$$

§ 44. *In any determinant the aggregate of the terms containing a particular element, say the element of the* p*th* *row and* q*th* *column, may be itself expressed as a determinant, viz. that obtained from the original determinant by changing into zero all the other elements of the* p*th* *row or of the* q*th* *column; and if the expression consisting of these terms be resolved into two factors, one of which is the element in question preceded by the sign* $(-1)^{p+q}$, *the co-factor may be expressed by the determinant of lower order obtained when the* p*th* *row and* q*th* *column of the original determinant have been deleted.*

Suppose the p^{th} row to be

$$e,\ g,\ a,\ \ldots,\ c,\ \beta,\ d,\ \ldots,\ h,\ f,$$

and let β be the particular element fixed on. Putting 0 for e in this row of the determinant, all the terms containing e will vanish, and, as none of them can also contain β, the terms containing β will remain. Similarly, putting 0 for g, we shall get rid of all the terms that contain g and shall not affect those which contain β. Thus we see that when we have made all the elements of the row 0 except β, there will still be left all the terms containing β. Moreover, no other terms will be left, for every term originally contained one of the elements of the row to which β belongs. Hence the first theorem is true.

Secondly, since the determinant we have reached has the elements of the p^{th} row all zero with the exception of one element which is β, if we change this β into 1 and place β as a multiplier outside the determinant form, we know (§ 26) that we get an equivalent expression. But then this new determinant, which has been found as the co-factor of β, has the element of the p^{th} row and q^{th} column 1 and all the other elements of the p^{th} row 0, therefore

(§ 43) it is equal to the determinant of lower order got by deleting the row and column in which the element 1 is found, provided we annex the sign-factor $(-1)^{p+q}$. Hence the co-factor of β is got by striking out of the original determinant the row and column in which β occurs and prefixing $(-1)^{p+q}$.

Example. In the expansion of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

the portion whose terms contain the element c_3 is equal to

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & c_3 & 0 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a_1 & a_2 & 0 & a_4 \\ b_1 & b_2 & 0 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & 0 & d_4 \end{vmatrix},$$

and therefore is equal to

$$c_3\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & 1 & 0 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \quad \text{or} \quad c_3\begin{vmatrix} a_1 & a_2 & 0 & a_4 \\ b_1 & b_2 & 0 & b_4 \\ c_1 & c_2 & 1 & c_4 \\ d_1 & d_2 & 0 & d_4 \end{vmatrix},$$

and consequently to

$$c_3\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ d_1 & d_2 & d_4 \end{vmatrix}.$$

§ 45. *A determinant of the* n*th order is expressible as the sum of* n *determinants, the first of which is obtained by changing into zero all the elements of any row or any column of the original determinant except the first element, the second by changing into zero all the elements of the same row or column except the second element, and so on.*

Let

$$a,\ b,\ c,\ d,\ \ldots\ldots$$

be the specified row of the original determinant, so that

$$\begin{array}{l} a,\ 0,\ 0,\ 0,\ \ldots\ldots \\ 0,\ b,\ 0,\ 0,\ \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

are the corresponding rows of the n determinants. Then the first of the n determinants equals that portion of the original determinant which includes all the terms containing a, the second equals that portion which includes all the terms containing b, and so on. But these portions make up the whole original determinant, for each term of it must contain one or other of the letters $a, b, c, d, \ldots\ldots$, and no term can contain two of them. Hence the theorem is true.

Example:—

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & 0 & b_4 \\ c_1 & c_2 & 0 & c_4 \\ d_1 & d_2 & 0 & d_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & 0 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & 0 & c_4 \\ d_1 & d_2 & 0 & d_4 \end{vmatrix}$$

$$+ \begin{vmatrix} a_1 & a_2 & 0 & a_4 \\ b_1 & b_2 & 0 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & 0 & d_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & 0 & a_4 \\ b_1 & b_2 & 0 & b_4 \\ c_1 & c_2 & 0 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix},$$

as is also evident from the addition-theorem of § 30.

§ 46. *A determinant of the* n*th order may be expressed as the aggregate of* n *products obtained by multiplying each element of any row by the determinant resulting from the deletion of the row and column to which the element belongs, the signs of the products being alternately + and –, and that of the first product + or – according as the number of the row is odd or even.*

Let the row taken be the p^{th}, its elements being in order

$$a,\ b,\ c,\ d,\ \ldots .$$

All the terms of the determinant which involve a may be expressed (§ 44) as the product of $(-1)^{p+1}a$ and the determinant obtained by deleting the p^{th} row and first column: similarly all the terms involving b may be expressed as the product of $(-1)^{p+2}b$ and the determinant obtained by deleting the p^{th} row and second column, and so on. But the aggregate of these terms is equal to the given determinant; and since the sign-factors

$$(-1)^{p+1},\ (-1)^{p+2},\ (-1)^{p+3},\ \ldots .$$

are alternately + and − 1, and the first of them + 1 or − 1 according as p is odd or even, the theorem is established.

EXAMPLES:—

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_3\begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - b_3\begin{vmatrix} a_1 & a_2 & a_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - d_3\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix};$$

and

$$\begin{aligned} |a_1b_2c_3d_4e_5| &= a_1|b_2c_3d_4e_5| - a_2|b_1c_3d_4e_5| + a_3|b_1c_2d_4e_5| - a_4|b_1c_2d_3e_5 + |a_5|b_1c_2d_3e_4|, \\ &= a_1|b_2c_3d_4e_5| - b_1|a_2c_3d_4e_5| + c_1|a_2b_3d_4e_5| - d_1|a_2b_3c_4e_5| + e_1|a_2b_3c_4d_5|, \\ &= -a_4|b_1c_2d_3e_5| + b_4|a_1c_2d_3e_5| - c_4|a_1b_2d_3e_5| + d_4|a_1b_2c_3e_5| - e_4|a_1b_2c_3d_5|, \\ &= \ldots\ldots \end{aligned}$$

§ 47. The number of terms in a determinant of the second order being 2, it follows from § 46 that the number of terms in a determinant of the third order is 2 × 3, that therefore the number of terms in a determinant of the fourth order is 2 × 3 × 4, and so generally, as in § 18. Similarly we have another proof of the theorem of § 23.

§ 48. *If the elements on one side of either diagonal of a determinant be all zero, the determinant consists of only one term, viz. the term composed of the elements of the said diagonal.*

EXAMPLE:—

$$\begin{vmatrix} 0 & 0 & 0 & d \\ 0 & 0 & c & g \\ 0 & b & f & i \\ a & e & h & j \end{vmatrix} = -a \begin{vmatrix} 0 & 0 & d \\ 0 & c & g \\ b & f & i \end{vmatrix},$$

$$= -ab \begin{vmatrix} 0 & d \\ c & g \end{vmatrix} = abcd.$$

§ 49. When the elements of a determinant are small letters and all different, the co-factor of any element is usually denoted by the corresponding capital letter, accompanied by the same suffix if suffixes occur. Thus in the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} \quad \text{or} \quad \Delta$$

the co-factor of f, which is $gb - ah$, is denoted by F. In this way fF stands for all the terms containing f, and we have

$$\begin{aligned} \Delta &= dD + eE + fF, \\ &= cC + fF + kK, \\ &= \ldots\ldots\ldots; \end{aligned}$$

and, generally,

$$\begin{aligned} |a_1 b_2 c_3 \ldots l_n| &= a_1 A_1 + a_2 A_2 + a_3 A_3 + \ldots + a_n A_n, \\ &= a_1 A_1 + b_1 B_1 + c_1 C_1 + \ldots + l_1 L_1, \\ &= \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

§ 50. Looking upon the determinant $|a_1 b_2 c_3 \ldots l_n|$, or $D(a_1 b_2 c_3 \ldots l_n)$, as a function of n^2 independent variables, viz. the elements, we have from the preceding by differentiation

$$\frac{\partial D}{\partial a_1} = A_1,$$

since all the terms of D which contain a_1 are included in $a_1 A_1$ and A_1 is independent of a_1. Similarly

$$\frac{\partial D}{\partial a_2} = A_2, \ \frac{\partial D}{\partial a_3} = A_3, \ldots.$$

$$\frac{\partial D}{\partial b_1} = B_1, \ \frac{\partial D}{\partial b_2} = B_2, \ldots.$$

We have thus a less arbitrary notation for the co-factors of the elements than that of § 49, and the identities there given may consequently be also put in the form

$$\begin{aligned} D(a_1 b_2 c_3 \ldots l_n) &= a_1 \frac{\partial D}{\partial a_1} + a_2 \frac{\partial D}{\partial a_2} + \ldots + a_n \frac{\partial D}{\partial a_n}, \\ &= a_1 \frac{\partial D}{\partial a_1} + b_1 \frac{\partial D}{\partial b_1} + \ldots + l_1 \frac{\partial D}{\partial l_1}, \\ &= \ldots\ldots\ldots\ldots\ldots \end{aligned}$$

§ 51. The result of § 46 will be easily seen to be of paramount importance in reference to the work of simplifying and expanding determinants, the finding of the expansion being made dependent upon the finding of the expansions of a number of determinants of the next lower order. Evidently also the advantage thus obtained will be augmented if by the use of previously established theorems we can succeed in changing several of the elements of a row of the original determinant into 0, for then the number of the said determinants of the next lower order will be correspondingly lessened. For example, consider the determinant

$$\begin{vmatrix} 10 & 4 & 17 & 13 \\ 4 & 2 & 8 & 6 \\ 3 & -1 & 8 & 1 \\ 7 & 5 & 20 & 17 \end{vmatrix},$$

and denote it by C. Then

$$\begin{aligned} C &= 10\begin{vmatrix} 2 & 8 & 6 \\ -1 & 8 & 1 \\ 5 & 20 & 17 \end{vmatrix} - 4\begin{vmatrix} 4 & 17 & 13 \\ -1 & 8 & 1 \\ 5 & 20 & 17 \end{vmatrix} + 3\begin{vmatrix} 4 & 17 & 13 \\ 2 & 8 & 6 \\ 5 & 20 & 17 \end{vmatrix} - 7\begin{vmatrix} 4 & 17 & 13 \\ 2 & 8 & 6 \\ -1 & 8 & 1 \end{vmatrix}, \\ &= 10\,(48) \quad -4\,(58) \quad +3\,(-4) \quad -7\,(16), \\ &= 124. \end{aligned}$$

Here we have at once applied the theorem of § 46.

Again

$$C = \begin{vmatrix} 2 & 4 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 5 & -1 & 12 & 4 \\ -3 & 5 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & 1 \\ 5 & 12 & 4 \\ -3 & 0 & 2 \end{vmatrix},$$

$$= 2\begin{vmatrix} 0 & 1 & 0 \\ -19 & 12 & -8 \\ -3 & 0 & 2 \end{vmatrix} = -2\begin{vmatrix} -19 & -8 \\ -3 & 2 \end{vmatrix},$$

$$= -2\begin{vmatrix} -31 & 0 \\ -3 & 2 \end{vmatrix} = 124.$$

Here we diminish each element of the first column by twice the corresponding element of the second column, each element of the third column by four times the corresponding element of the second column, and each element of the fourth column by three times the corresponding element of the second column, the result being a determinant with a row containing three zero elements, from which by means of the theorem of § 46 or of § 41 we pass to a single determinant of the next lower order; then this determinant is treated in similar fashion; and so on.

It will be observed that one of the elements of the second row of C is a measure of each of the other elements of the row, and that to this peculiarity is due the possibility of transforming C, as above, into a determinant with a row containing three zero elements. The second column possesses the same peculiarity, so that we might also proceed as follows:—

$$C = \begin{vmatrix} 22 & 0 & 49 & 17 \\ 10 & 0 & 24 & 8 \\ 3 & -1 & 8 & 1 \\ 22 & 0 & 60 & 22 \end{vmatrix} = \begin{vmatrix} 22 & 49 & 17 \\ 10 & 24 & 8 \\ 22 & 60 & 22 \end{vmatrix},$$

$$= \begin{vmatrix} 5 & -2 & 17 \\ 2 & 0 & 8 \\ 0 & -6 & 22 \end{vmatrix} = -204 + 88 + 240,$$

$$= 124.$$

Had such not been the case we could first have transformed C into a determinant having the peculiarity referred to, *e.g.*, a determinant having one of its elements 1; and in this way the second mode of procedure can be seen to be always possible.

Example. Show that

$$\begin{vmatrix} b_1 & a & a & a \\ a & b_2 & a & a \\ a & a & b_3 & a \\ a & a & a & b_4 \end{vmatrix} = (b_1-a)(b_2-a)(b_3-a)(b_4-a)\left\{1+\frac{a}{b_1-a}+\frac{a}{b_2-a}+\frac{a}{b_3-a}+\frac{a}{b_4-a}\right\}.$$

It is clear (§ 42) that the given determinant is equal to

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & b_1 & a & a & a \\ 0 & a & b_2 & a & a \\ 0 & a & a & b_3 & a \\ 0 & a & a & a & b_4 \end{vmatrix}.$$

Multiplying each element of the first row of this determinant by a, and subtracting the result from the corresponding element of each of the other rows, we have

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -a & b_1-a & 0 & 0 & 0 \\ -a & 0 & b_2-a & 0 & 0 \\ -a & 0 & 0 & b_3-a & 0 \\ -a & 0 & 0 & 0 & b_4-a \end{vmatrix}.$$

This we know (§ 46) is equal to

$$\begin{vmatrix} b_1-a & 0 & 0 & 0 \\ 0 & b_2-a & 0 & 0 \\ 0 & 0 & b_3-a & 0 \\ 0 & 0 & 0 & b_4-a \end{vmatrix} + a\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b_2-a & 0 & 0 \\ 0 & 0 & b_3-a & 0 \\ 0 & 0 & 0 & b_4-a \end{vmatrix} - a\begin{vmatrix} 1 & 1 & 1 & 1 \\ b_1-a & 0 & 0 & 0 \\ 0 & 0 & b_3-a & 0 \\ 0 & 0 & 0 & b_4-a \end{vmatrix}$$

$$+ a\begin{vmatrix} 1 & 1 & 1 & 1 \\ b_1-a & 0 & 0 & 0 \\ 0 & b_2-a & 0 & 0 \\ 0 & 0 & 0 & b_4-a \end{vmatrix} - a\begin{vmatrix} 1 & 1 & 1 & 1 \\ b_1-a & 0 & 0 & 0 \\ 0 & b_2-a & 0 & 0 \\ 0 & 0 & b_3-a & 0 \end{vmatrix}.$$

Here the first two determinants have all their elements 0 on one side of the principal diagonal, and the others have each a column with all the elements 0 except one, hence we transform the expression into

$$(b_1-a)(b_2-a)(b_3-a)(b_4-a) + a(b_2-a)(b_3-a)(b_4-a) + a\begin{vmatrix} b_1-a & 0 & 0 \\ 0 & b_3-a & 0 \\ 0 & 0 & b_4-a \end{vmatrix}$$

$$+ a\begin{vmatrix} b_1-a & 0 & 0 \\ 0 & b_2-a & 0 \\ 0 & 0 & b_4-a \end{vmatrix} + a\begin{vmatrix} b_1-a & 0 & 0 \\ 0 & b_2-a & 0 \\ 0 & 0 & b_3-a \end{vmatrix},$$

whence readily comes the result desired.

In the preceding process, the first step, viz. taking an equivalent determinant of higher order, is worthy of the learner's attention, as being conducive to symmetry in obtaining the result in the particular form wanted.

EXERCISES. SET IX.

Find the single numbers to which the following determinants are equal:—

$$1.\ \begin{vmatrix} 3 & 1 & 4 & 1 \\ 2 & 2 & 8 & 5 \\ 1 & 6 & 4 & 2 \\ 3 & 2 & 5 & 3 \end{vmatrix}. \qquad 2.\ \begin{vmatrix} 3 & 7 & 4 & 3 \\ 7 & 4 & 3 & 5 \\ 2 & 1 & 9 & 4 \\ 8 & 6 & 4 & 7 \end{vmatrix}. \qquad 3.\ \begin{vmatrix} 2 & 1 & 3 & 4 \\ 7 & 4 & 5 & 9 \\ 3 & 3 & 6 & 2 \\ 1 & 7 & 7 & 5 \end{vmatrix}.$$

$$4.\ \begin{vmatrix} 8 & 7 & 5 & 10 \\ 4 & 3 & 9 & 2 \\ 8 & 9 & 6 & 12 \\ 3 & 1 & 2 & 4 \end{vmatrix}. \qquad 5.\ \begin{vmatrix} 10 & 8 & 9 & 14 \\ 17 & 15 & 18 & 11 \\ 15 & 19 & 10 & 13 \\ 16 & 17 & 18 & 10 \end{vmatrix}. \qquad 6.\ \begin{vmatrix} 21 & -22 & 6 & 14 \\ 12 & 13 & -7 & 18 \\ 25 & 14 & 18 & -26 \\ -7 & 17 & -12 & 4 \end{vmatrix}.$$

$$7.\ \begin{vmatrix} 3 & 1 & 5 & 4 & 2 \\ 7 & 6 & 4 & 1 & 3 \\ 1 & 3 & 2 & 9 & 4 \\ 2 & 2 & 9 & 2 & 1 \\ 8 & 6 & 1 & 3 & 4 \end{vmatrix}. \qquad 8.\ \begin{vmatrix} 5 & -1 & 4 & 6 & -2 \\ -1 & 4 & 6 & -2 & 5 \\ 4 & 6 & -2 & 5 & -1 \\ 6 & -2 & 5 & -1 & 4 \\ -2 & 5 & -1 & 4 & 6 \end{vmatrix}.$$

$$9.\ \begin{vmatrix} 2 & 4 & 3 & 1 & 4 & 3 \\ -4 & 2 & -3 & 2 & -1 & 2 \\ 5 & -1 & 6 & 2 & -1 & 5 \\ 1 & 1 & 1 & -2 & -2 & -2 \\ 7 & -3 & -5 & 1 & 4 & 2 \\ 3 & 1 & 2 & -1 & 2 & 3 \end{vmatrix}. \qquad 10.\ \begin{vmatrix} 12 & 22 & 14 & 17 & 20 & 10 \\ 16 & -4 & 7 & 1 & -2 & 15 \\ 10 & -3 & -2 & 3 & -2 & 8 \\ 7 & 12 & 8 & 9 & 11 & 6 \\ 11 & 2 & 4 & -8 & 1 & 9 \\ 24 & 6 & 6 & 3 & 4 & 22 \end{vmatrix}.$$

Find the ordinary expansion of the following determinants:-

$$11.\ \begin{vmatrix} 0 & 0 & k & l & x \\ 0 & 0 & h & x & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & x & e & f & g \\ x & a & b & c & d \end{vmatrix}. \qquad 12.\ \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ 0 & b_2 & 0 & 0 & 0 \\ 0 & b_3 & c_3 & d_3 & e_3 \\ 0 & b_4 & 0 & d_4 & 0 \\ 0 & b_5 & 0 & d_5 & e_5 \end{vmatrix}.$$

$$13.\ \begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}. \qquad 14.\ \begin{vmatrix} 0 & d & d & d \\ a & 0 & a & a \\ b & b & 0 & b \\ c & c & c & 0 \end{vmatrix}. \qquad 15.\ \begin{vmatrix} 0 & a & b & c \\ a & 0 & \gamma & \beta \\ b & \gamma & 0 & a \\ c & \beta & a & 0 \end{vmatrix}.$$

16. $\begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix}.$ 17. $\begin{vmatrix} 1 & y & 0 & 0 \\ 1 & x & y & 0 \\ 1 & 0 & x & y \\ 1 & 0 & 0 & x \end{vmatrix}.$ 18. $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & b+c & a & a \\ 1 & b & c+a & b \\ 1 & c & c & a+b \end{vmatrix}.$

19. $\begin{vmatrix} a & b & c & d \\ -a & b & x & y \\ -a & -b & c & z \\ -a & -b & -c & d \end{vmatrix}.$ 20. $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}.$ 21. $\begin{vmatrix} a+b & b & c & d \\ a & b+c & c & d \\ a & b & c+d & d \\ a & b & c & d+e \end{vmatrix}.$

22. $\begin{vmatrix} x & 0 & 0 & 0 \dots\dots & 0 & a_n \\ -1 & x & x^2 & x^3 \dots\dots & x^{n-1} & a_{n-1} \\ 0 & -1 & 0 & 0 \dots\dots & 0 & a_{n-2} \\ 0 & 0 & -1 & 0 \dots\dots & 0 & a_{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \dots\dots & 0 & a_1 \\ 0 & 0 & 0 & 0 \dots\dots & -1 & a_0 \end{vmatrix}.$ 23. $\begin{vmatrix} a_1 & b_2 & 0 & 0 \dots\dots & 0 & 0 \\ a_2 & -b_1 & b_3 & 0 \dots\dots & 0 & 0 \\ a_3 & 0 & -b_2 & b_4 \dots\dots & 0 & 0 \\ a_4 & 0 & 0 & -b_3 \dots\dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & 0 & 0 & 0 \dots\dots & -b_{n-1} & b_{n+1} \\ a_{n+1} & 0 & 0 & 0 \dots\dots & 0 & -b_n \end{vmatrix}.$

Establish the following identities:—

24. $$\begin{vmatrix} a^2+1 & ab & ac & ad \\ ab & b^2+1 & bc & bd \\ ac & bc & c^2+1 & cd \\ ad & bd & cd & d^2+1 \end{vmatrix} = a^2+b^2+c^2+d^2+1.$$

25. $$\begin{vmatrix} a+b+c & d & d & d \\ a & b+c+d & a & a \\ b & b & c+d+a & b \\ c & c & c & d+a+b \end{vmatrix} = 4\Sigma a^2bc.$$

26. $$\begin{vmatrix} a+x & a-x & a-y & a+y \\ a-x & a-y & a+y & a+x \\ a-y & a+y & a+x & a-x \\ a+y & a+x & a-x & a-y \end{vmatrix} = -16a(x-y)(x+y)^2.$$

27. $$\begin{vmatrix} 1 & a & a & a^2 \\ 1 & b & b & b^2 \\ 1 & c & c' & cc' \\ 1 & d & d' & dd' \end{vmatrix} = (a-b)\begin{vmatrix} 1 & ab & a+b \\ 1 & cd' & c+d' \\ 1 & c'd & c'+d \end{vmatrix}.$$

28. Resolve into simple factors

$$\begin{vmatrix} x & a_1 & a_2 & a_3 & 1 \\ a_1 & x & a_2 & a_3 & 1 \\ a_1 & a_2 & x & a_3 & 1 \\ a_1 & a_2 & a_3 & x & 1 \\ a_1 & a_2 & a_3 & a_4 & 1 \end{vmatrix}.$$

29. Without finding the ordinary expansión of the determinants, show that

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_6 \\ 1 & a_1 & a_2 & a_5 \\ 0 & 1 & a_1 & a_4 \\ 0 & 0 & 1 & a_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_4 & a_5 \\ 1 & a_1 & a_3 & a_4 \\ 0 & 1 & a_2 & a_3 \\ 0 & 0 & a_1 & a_2 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 & a_4 & a_6 \\ 1 & a_1 & a_2 & a_4 \\ 0 & 1 & a_1 & a_3 \\ 0 & 0 & 1 & a_2 \end{vmatrix}.$$

30. Find the complete differential of

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

due to the independent variation of all the elements.

§ 52. *If the first element of a row of a determinant be multiplied by the co-factor of the first element of another row, the second element of the former row be multiplied by the co-factor of the second element of the latter, and so on, the sum of the products is equal to zero.*

Let the determinant be

$$\begin{vmatrix} \cdots\cdots\cdots\cdots\cdots \\ a, b, c, d, \ldots\ldots \\ \cdots\cdots\cdots\cdots\cdots \\ p, q, r, s, \ldots\ldots \\ \cdots\cdots\cdots\cdots\cdots \end{vmatrix},$$

and let the two rows whose elements are here partially given be the two rows referred to. Then using the notation of § 49 we have

$$pP + qQ + rR + sS + \dots = \left|\begin{array}{l} \dots\dots\dots\dots\dots \\ a,\ b,\ c,\ d,\ \dots\dots \\ \dots\dots\dots\dots\dots \\ p,\ q,\ r,\ s,\ \dots\dots \\ \dots\dots\dots\dots\dots \end{array}\right|.$$

Here P, being in magnitude equal to the determinant got by deleting the row and column to which p belongs, must be independent of $p, q, r, s, \dots$, and therefore suffer no change from any change of these elements; and the same is in like manner true of $Q, R, S, \dots$. Hence, changing p into a, q into b, r into c, and so on, in both members of the preceding identity, we have

$$\begin{aligned} aP + bQ + cR + dS + \dots &= \left|\begin{array}{l} \dots\dots\dots\dots\dots \\ a,\ b,\ c,\ d,\ \dots\dots \\ \dots\dots\dots\dots\dots \\ a,\ b,\ c,\ d,\ \dots\dots \\ \dots\dots\dots\dots\dots \end{array}\right|, \\ &= 0, \qquad\qquad (\S\ 27) \end{aligned}$$

as was to be proved.

§ 53. *If the first pair of elements in the first row of a determinant be taken in succession with every pair below it, and the determinants of the second degree which have these pairs for rows be placed in order as the elements of the first column of a new determinant, and if the like be done in the case of the second and following pairs of* *consecutive elements in the row, then the new determinant thus obtained divided by the product of all the elements of the first row of the original determinant except the first and last is equal to the original determinant.*

Let the given determinant be

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \dots k_1 & l_1 \\ a_2 & b_2 & c_2 & d_2 \dots k_2 & l_2 \\ a_3 & b_3 & c_3 & d_3 \dots k_3 & l_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & d_n \dots k_n & l_n \end{vmatrix} \quad \text{or} \quad \Delta.$$

Multiplying each element of the first column by $-b_1$ and adding to the result a_1 times the corresponding element of the second column, we have (§ 30)

$$-b_1\Delta = \begin{vmatrix} 0 & b_1 & c_1 & d_1 \dots k_1 & l_1 \\ -a_2b_1 + b_2a_1 & b_2 & c_2 & d_2 \dots k_2 & l_2 \\ -a_3b_1 + b_3a_1 & b_3 & c_3 & d_3 \dots k_3 & l_3 \\ \dots & \dots & \dots & \dots & \dots \\ -a_nb_1 + b_na_1 & b_n & c_n & d_n \dots k_n & l_n \end{vmatrix}.$$

Again, multiplying each element of the second column of this determinant by $-c_1$ and adding to the result b_1 times the corresponding element of the third column, we have

$$(-1)^2b_1c_1\,\Delta = \begin{vmatrix} 0 & 0 & c_1 & d_1 \dots k_1 & l_1 \\ -a_2b_1 + b_2a_1 & -b_2c_1 + c_2b_1 & c_2 & d_2 \dots k_2 & l_2 \\ -a_3b_1 + b_3a_1 & -b_3c_1 + c_3b_1 & c_3 & d_3 \dots k_3 & l_3 \\ \dots & \dots & \dots & \dots & \dots \\ -a_nb_1 + b_na_1 & -b_nc_1 + c_nb_1 & c_n & d_n \dots k_n & l_n \end{vmatrix}.$$

This process being continued, the final result is

$$(-1)^{n-1}b_1c_1d_1\ldots l_1\,\Delta$$

$$=\begin{vmatrix} 0 & 0 & 0 & \ldots & 0 & l_1 \\ -a_2b_1+b_2a_1 & -b_2c_1+c_2b_1 & -c_2d_1+d_2c_1 & \ldots & -k_2l_1+l_2k_1 & l_2 \\ -a_3b_1+b_3a_1 & -b_3c_1+c_3b_1 & -c_3d_1+d_3c_1 & \ldots & -k_3l_1+l_3k_1 & l_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_nb_1+b_na_1 & -b_nc_1+c_nb_1 & -c_nd_1+d_nc_1 & \ldots & -k_nl_1+l_nk_1 & l_n \end{vmatrix},$$

$$=\begin{vmatrix} -a_2b_1+b_2a_1 & -b_2c_1+c_2b_1 & -c_2d_1+d_2c_1 & \ldots & -k_2l_1+l_2k_1 \\ -a_3b_1+b_3a_1 & -b_3c_1+c_3b_1 & -c_3d_1+d_3c_1 & \ldots & -k_3l_1+l_3k_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_nb_1+b_na_1 & -b_nc_1+c_nb_1 & -c_nd_1+d_nc_1 & \ldots & -k_nl_1+l_nk_1 \end{vmatrix}(-1)^{n-1}l_1;$$

and dividing by $(-1)^{n-1}b_1c_1d_1\ldots l_1$, we obtain

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \ldots l_1 \\ a_2 & b_2 & c_2 & d_2 \ldots l_2 \\ a_3 & b_3 & c_3 & d_3 \ldots l_3 \\ \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & d_n \ldots l_n \end{vmatrix} = \frac{1}{b_1c_1d_1\ldots k_1}\begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, & \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \ldots & \begin{vmatrix} k_1 & l_1 \\ k_2 & l_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, & \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \ldots & \begin{vmatrix} k_1 & l_1 \\ k_3 & l_3 \end{vmatrix} \\ \cdots & \cdots & \cdots \\ \begin{vmatrix} a_1 & b_1 \\ a_n & b_n \end{vmatrix}, & \begin{vmatrix} b_1 & c_1 \\ b_n & c_n \end{vmatrix}, \ldots & \begin{vmatrix} k_1 & l_1 \\ k_n & l_n \end{vmatrix} \end{vmatrix},$$

as was to be proved.

§ 54. The new determinant found in the preceding paragraph being one degree lower than the original, the theorem is important as affording an easy means of evaluating a determinant whose elements are expressed in figures. Thus, taking the example already dealt with (§ 47), we have

$$\begin{vmatrix} 10 & 4 & 3 & 7 \\ 4 & 2 & -1 & 5 \\ 17 & 8 & 8 & 20 \\ 13 & 6 & 1 & 17 \end{vmatrix} = \frac{1}{12}\begin{vmatrix} 4 & -10 & 22 \\ 12 & 8 & 4 \\ 8 & -14 & 44 \end{vmatrix},$$

$$= \frac{4}{3}\begin{vmatrix} 2 & -5 & 11 \\ 3 & 2 & 1 \\ 4 & -7 & 22 \end{vmatrix},$$

$$= -\frac{4}{15}\begin{vmatrix} 19 & -27 \\ 6 & -33 \end{vmatrix},$$

$$= -\frac{4}{5}(-209 + 54),$$

$$= 124.$$

Here, looking at the first two rows of the given determinant, we at once mentally evaluate

$$\begin{vmatrix} 10 & 4 \\ 4 & 2 \end{vmatrix}, \quad \begin{vmatrix} 4 & 3 \\ 2 & -1 \end{vmatrix}, \quad \begin{vmatrix} 3 & 7 \\ -1 & 5 \end{vmatrix},$$

and place the results 4, -10, 22 for the first row of the new determinant; similarly we proceed with the first and third rows, and with the first and fourth rows. This gives us a determinant of the third degree, from which we remove the factors 2, 4, 2; then we treat the resulting determinant as the first was treated, and thus have at last only to deal with a determinant of the second degree.

§ 55. If one of the elements included between the first and last of the first row be zero, the theorem is clearly inapplicable; but then it has to be remembered in such a case that any row or any column may be made the first row.

EXAMPLE 1. Find the values of x_1, x_2, x_3, ..., x_n which satisfy the n equations

$$\left.\begin{array}{l} a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = a_0 \\ b_1x_1 + b_2x_2 + b_3x_3 + \ldots + b_nx_n = b_0 \\ c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n = c_0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ l_1x_1 + l_2x_2 + l_3x_3 + \ldots + l_nx_n = l_0 \end{array}\right\} .$$

Looking back to the special cases of this problem which are solved in § 9, we see that the process consisted in multiplying both sides of the first equation by a certain number, both sides of the second equation by a second number, and so on, and then taking the sum of the left-hand members of the equations thus resulting, and equating it to the sum of the right-hand members, when it was found that the final equation was free of all the unknown quantities except one, whose value could therefore at once be given.

If this process can be generalized, the first requisite is to find, if possible, what multipliers will be suitable. To do this, let us for the present denote the multipliers in order by

$$\mu_1,\ \mu_2,\ \mu_3,\ \ldots\ldots,\ \mu_n.$$

Then performing the multiplications and additions referred to, we have

$$\left.\begin{array}{l} (\mu_1a_1 + \mu_2b_1 + \mu_3c_1 + \ldots + \mu_nl_1)\,x_1 \\ + (\mu_1a_2 + \mu_2b_2 + \mu_3c_2 + \ldots + \mu_nl_2)\,x_2 \\ + (\mu_1a_3 + \mu_2b_3 + \mu_3c_3 + \ldots + \mu_nl_3)\,x_3 \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + (\mu_1a_n + \mu_2b_n + \mu_3c_n + \ldots + \mu_nl_n)\,x_n \end{array}\right\} = \mu_1a_0 + \mu_2b_0 + \mu_3c_0 + \ldots + \mu_nl_0\,;$$

and what we wish to know is whether there be values of μ_1, μ_2, μ_3, ... which will make all the coefficients of the unknown quantities in this equation vanish except one. Now, looking to the determinant

$$\left|\begin{array}{llll} a_1 & a_2 & a_3 \ldots. & a_n \\ b_1 & b_2 & b_3 \ldots. & b_n \\ c_1 & c_2 & c_3 \ldots. & c_n \\ \ldots\ldots & \ldots & \ldots\ldots & \ldots \\ l_1 & l_2 & l_3 \ldots. & l_n \end{array}\right| \quad \text{or} \quad \Delta,$$

whose elements are in order the coefficients of the unknown quantities in the original equations, and observing that

$$a_2,\ b_2,\ c_2,\ \ldots.,\ l_2$$

constitute a column, it is readily seen (§ 50) that the coefficient of x_2, viz.

$$\mu_1a_2 + \mu_2b_2 + \mu_3c_2 + \ldots. + \mu_nl_2$$

would vanish if we took

$$\begin{aligned} \mu_1 &= \text{the co-factor of } a_1 \text{ in } \Delta, \\ \mu_2 &= \ldots\ldots\ldots\ldots\ldots\ldots\ \ b_1\ \ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mu_n &= \ldots\ldots\ldots\ldots\ldots\ldots\ \ l_1\ \ldots\ldots \end{aligned}$$

and that for the like reason the coefficients of x_3, x_4, ..., x_n would also vanish, while (§ 46) the same could not be said of the coefficient of x_1, which in fact would be Δ.

Taking therefore in order the co-factors of a_1, b_1, c_1, ..., l_1 as our multipliers (which is what was done in the particular cases of § 9), the equation which results after the addition is

$$\begin{vmatrix} a_1 & a_2 & a_3 \ldots\ldots & a_n \\ b_1 & b_2 & b_3 \ldots\ldots & b_n \\ c_1 & c_2 & c_3 \ldots\ldots & c_n \\ \ldots & \ldots & \ldots & \ldots \\ l_1 & l_2 & l_3 \ldots\ldots & l_n \end{vmatrix} x_1 = \begin{vmatrix} a_0 & a_2 & a_3 \ldots\ldots & a_n \\ b_0 & b_2 & b_3 \ldots\ldots & b_n \\ c_0 & c_2 & c_3 \ldots\ldots & c_n \\ \ldots & \ldots & \ldots & \ldots \\ l_0 & l_2 & l_3 \ldots\ldots & l_n \end{vmatrix},$$

$$\text{and } \therefore \qquad x_1 = \frac{|a_0 b_2 c_3 \ldots\ldots l_n|}{|a_1 b_2 c_3 \ldots\ldots l_n|}.$$

If in the original equations we were to write the terms containing x_2 before the terms containing x_1 we should have in like manner

$$\begin{vmatrix} a_2 & a_1 & a_3 \ldots\ldots & a_n \\ b_2 & b_1 & b_3 \ldots\ldots & b_n \\ c_2 & c_1 & c_3 \ldots\ldots & c_n \\ \ldots & \ldots & \ldots & \ldots \\ l_2 & l_1 & l_3 \ldots\ldots & l_n \end{vmatrix} x_2 = \begin{vmatrix} a_0 & a_1 & a_3 \ldots\ldots & a_n \\ b_0 & b_1 & b_3 \ldots\ldots & b_n \\ c_0 & c_1 & c_3 \ldots\ldots & c_n \\ \ldots & \ldots & \ldots & \ldots \\ l_0 & l_1 & l_3 \ldots\ldots & l_n \end{vmatrix},$$

and, by transposing the first two columns on both sides of the equation, it would follow that

$$x_2 = \frac{|a_1 b_0 c_3 \ldots\ldots l_n|}{|a_1 b_2 c_3 \ldots\ldots l_n|}.$$

$$\text{Similarly} \quad x_3 = \frac{|a_1 b_2 c_0 \ldots\ldots l_n|}{|a_1 b_2 c_3 \ldots\ldots l_n|},$$

and so on.

EXAMPLE 2. If the $n+1$ statements

$$\left.\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 + \ldots\ldots + a_nx_n &= a_0 \\ b_1x_1 + b_2x_2 + b_3x_3 + \ldots\ldots + b_nx_n &= b_0 \\ c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots + c_nx_n &= c_0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ l_1x_1 + l_2x_2 + l_3x_3 + \ldots\ldots + l_nx_n &= l_0 \\ m_1x_1 + m_2x_2 + m_3x_3 + \ldots\ldots + m_nx_n &= m_0 \end{aligned}\right\}$$

in regard to the n quantities $x_1, x_2, x_3, \dots, x_n$ be true, find what relation must exist among the other quantities mentioned.

From the first n equations we are able to show, as is done above, that $x_1, x_2, \dots, x_n$ are equal to

$$\frac{|a_0b_2c_3\dots l_n|}{|a_1b_2c_3\dots l_n|},\quad \frac{|a_1b_0c_3\dots l_n|}{|a_1b_2c_3\dots l_n|},\ \dots\dots,\quad \frac{|a_1b_2c_3\dots l_0|}{|a_1b_2c_3\dots l_n|},\ \dots\dots$$

Consequently from substitution in the $(n+1)^{\text{th}}$ equation it follows that

$$m_1\frac{|a_0b_2c_3\dots l_n|}{|a_1b_2c_3\dots l_n|} + m_2\frac{|a_1b_0c_3\dots l_n|}{|a_1b_2c_3\dots l_n|} + \dots\dots + m_n\frac{|a_1b_2c_3\dots l_0|}{|a_1b_2c_3\dots l_n|} = m_0 \ \dots\dots\dots \ (1),$$

which is the relation desired.

The learner may already know that as this statement logically results from the $n+1$ given statements regarding $x_1, x_2, \dots\dots, x_n$, and does not itself involve $x_1, x_2, \dots\dots, x_n$, we are said to have *eliminated* $x_1, x_2, \dots\dots, x_n$, and that the concluding statement referred to is called the RESULTANT *of the elimination*, or the *resultant of the set of equations*.

The resultant (1) can however be put in simpler form. We have from it

$$m_1|a_0b_2c_3\dots l_n| + m_2|a_1b_0c_3\dots l_n| + \dots + m_n|a_1b_2c_3\dots l_0| = m_0|a_1b_2c_3\dots l_n|;$$

therefore by transposition of columns

$$(-1)^{n-1}m_1|a_2b_3c_4\dots l_0| + (-1)^{n-2}m_2|a_1b_3c_4\dots l_0| + \dots + m_n|a_1b_2c_3\dots l_0| - m_0|a_1b_2c_3\dots l_n| = 0;$$

and consequently (§ 46)

$$\begin{vmatrix} a_1 & a_2 & a_3 \dots & a_n & a_0 \\ b_1 & b_2 & b_3 \dots & b_n & b_0 \\ c_1 & c_2 & c_3 \dots & c_n & c_0 \\ \dots & \dots & \dots & \dots & \dots \\ l_1 & l_2 & l_3 \dots & l_n & l_0 \\ m_1 & m_2 & m_3 \dots & m_n & m_0 \end{vmatrix} = 0.$$

EXAMPLE 3. If the n statements

$$\left.\begin{array}{l} a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = 0 \\ b_1x_1 + b_2x_2 + b_3x_3 + \dots + b_nx_n = 0 \\ c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ l_1x_1 + l_2x_2 + l_3x_3 + \dots + l_nx_n = 0 \end{array}\right\}$$

in regard to the n quantities $x_1, x_2, x_3, \dots, x_n$ be true, and $x_1, x_2, x_3, \dots, x_n$ be not each equal to zero, find what relation must exist among the coefficients.

Dividing both members of each equation by x_n we have n statements in regard to $n-1$ quantities $\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n}$, and thus by means of what immediately precedes we have as the desired relation

$$\begin{vmatrix} a_1 & a_2 & a_3 \ldots\ldots a_n \\ b_1 & b_2 & b_3 \ldots\ldots b_n \\ c_1 & c_2 & c_3 \ldots\ldots c_n \\ \ldots & \ldots & \ldots\ldots\ldots \\ l_1 & l_3 & l_3 \ldots\ldots l_n \end{vmatrix} = 0.$$

EXERCISES. SET X.

Use the process of § 54 to find the simplest equivalent of each of the determinants—

1. $\begin{vmatrix} 3 & 4 & 1 & 2 \\ 6 & 9 & 7 & 5 \\ 7 & 10 & 5 & 8 \\ 4 & 2 & 9 & 3 \end{vmatrix}$.

2. $\begin{vmatrix} 7 & 3 & 9 & 4 \\ 3 & 9 & 4 & 7 \\ 2 & 6 & 1 & 5 \\ 10 & 7 & 14 & 10 \end{vmatrix}$.

3. $\begin{vmatrix} 3 & 4 & 7 & 2 & 5 \\ -3 & 1 & 2 & 5 & -1 \\ 6 & -2 & 3 & -1 & 4 \\ 5 & 9 & -2 & 3 & 2 \\ 1 & -3 & 5 & 3 & 7 \end{vmatrix}$.

4. $\begin{vmatrix} 16 & 14 & 12 & 9 & 15 \\ 5 & 4 & 3 & 1 & 6 \\ 2 & 8 & 4 & 9 & 7 \\ 3 & -2 & -2 & -3 & -6 \\ 1 & 3 & 2 & 3 & 2 \end{vmatrix}$.

With the help of determinants solve the following sets of equations:—

5. $\left.\begin{aligned} 4x+7y+3z-2w &= 9 \\ 2x-y-4z+3w &= 13 \\ 3x+2y-7z-4w &= 2 \\ 5x-3y+z+5w &= 13 \end{aligned}\right\}$.

6. $\left.\begin{aligned} 3x+2y+4z-w &= 13 \\ 5x+y-z+2w &= 9 \\ 2x+3y-7z+3w &= 14 \\ 4x-4y+3z-5w &= 4 \end{aligned}\right\}$.

7. $\left.\begin{aligned} v+w-y &= a \\ w+x-z &= b \\ x+y-v &= c \\ y+z-w &= d \\ z+v-x &= e \end{aligned}\right\}$.

8. $\left.\begin{aligned} v+w+x-y &= a \\ w+x+y-z &= b \\ x+y+z-v &= c \\ y+z+v-w &= d \\ z+v+w-x &= e \end{aligned}\right\}$.

9. $\left.\begin{aligned} w+x+y+z &= 1 \\ aw+bx+cy+dz &= e \\ a^2w+b^2x+c^2y+d^2z &= e^2 \\ a^3w+b^3x+c^3y+d^3z &= e^3 \end{aligned}\right\}$.

10. $\left.\begin{aligned} w+x+y+z &= 1 \\ w+ax+by+cz &= 0 \\ w+a^2x+b^2y+c^2z &= 0 \\ w+a^3x+b^3y+c^3z &= 0 \end{aligned}\right\}$.

11. $$\left.\begin{aligned} v - 2w - 2x + y + 3z &= a \\ w - 2x - 2y + z + 3v &= b \\ x - 2y - 2z + v + 3w &= c \\ y - 2z - 2v + w + 3x &= d \\ z - 2v - 2w + x + 3y &= e \end{aligned}\right\} .$$

12. What relation must exist between a, b, c, d if the equations

$$\begin{aligned} ax + by + cz + d &= 0, \\ bx + ay + dz + c &= 0, \\ ax + cy + bz + d &= 0, \\ cx + ay + dz + b &= 0, \end{aligned}$$

be simultaneously true?

13. If the equations

$$\begin{aligned} a_1x^3 + b_1x^2 + c_1x + d_1 &= 0, \\ a_1x^4 + b_1x^3 + c_1x^2 + d_1x \quad &= 0, \\ b_2x^2 + c_2x + d_2 &= 0, \\ b_2x^3 + c_2x^2 + d_2x \quad &= 0, \\ b_2x^4 + c_2x^3 + d_2x^2 \quad &= 0, \end{aligned}$$

be simultaneously true (which evidently will be the case if the first and third be simultaneously true, *i.e.* have a common root), find the relation which must exist between a_1, b_1, c_1, d_1, b_2, c_2, d_2.

Similarly find the resultant in the case of each of the following pairs of equations:—

14. $$\left.\begin{aligned} a_1x^2 + b_1x + c_1 &= 0 \\ a_2x^2 + b_2x + c_2 &= 0 \end{aligned}\right\} .$$

15. $$\left.\begin{aligned} a_1x^4 + b_1x^3 + c_1x^2 + d_1x + e_1 &= 0 \\ c_2x^2 + d_2x + e_2 &= 0 \end{aligned}\right\} .$$

16. $$\left.\begin{aligned} a_1x^3 + b_1x^2 + c_1x + d_1 &= 0 \\ a_2x^3 + b_2x^2 + c_2x + d_2 &= 0 \end{aligned}\right\} .$$

17. $$\left.\begin{aligned} a_1x^4 + b_1x^3 + c_1x^2 + d_1x + e_1 &= 0 \\ b_2x^3 + c_2x^2 + d_2x + e_2 &= 0 \end{aligned}\right\} .$$

18. $$\left.\begin{aligned} a_1x^4 + c_1x^2 + d_1x + e_1 &= 0 \\ a_2x^4 + b_2x^3 + d_2x + e_2 &= 0 \end{aligned}\right\} .$$

19. $$\left.\begin{aligned} a_1x^6 + b_1x^2 + c_1 &= 0 \\ d_1x^4 + e_1x^2 + f_1 &= 0 \end{aligned}\right\} .$$

20. Prove that

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \ldots\ldots & a_n \\ b_1 & b_2 & b_3 & b_4 \ldots\ldots & b_n \\ c_1 & c_2 & c_3 & c_4 \ldots\ldots & c_n \\ d_1 & d_2 & d_3 & d_4 \ldots\ldots & d_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ l_1 & l_2 & l_3 & l_4 \ldots\ldots & l_n \end{vmatrix} = \frac{1}{a_1{}^{n-2}} \begin{vmatrix} |a_1 b_2| & |a_1 b_3| & |a_1 b_4| \ldots\ldots & |a_1 b_n| \\ |a_1 c_2| & |a_1 c_3| & |a_1 c_4| \ldots\ldots & |a_1 c_n| \\ |a_1 d_2| & |a_1 d_3| & |a_1 d_4| \ldots\ldots & |a_1 d_n| \\ \cdots & \cdots & \cdots & \cdots \\ |a_1 l_2| & |a_1 l_3| & |a_1 l_4| \ldots\ldots & |a_1 l_n| \end{vmatrix} .$$

§ 56. The introduction of letters with a suffix, like a_1, c_5, &c., increases many fold the stock of algebraical symbols; and the new symbols have the special advantage that each one indicates that the number denoted by it belongs to a certain *series* of numbers, and also shows which one of the series is intended. These advantages have been apparent in what has preceded; but there is an extension of the same notation from which we have symbols still more appropriate for denoting the numbers which are the elements of a determinant. This consists in making use of *a pair of suffixes*, written one after the other, with or without a comma between them, thus,

$$a_{1,2},\ a_{34},\ \ldots\ldots;$$

$a_{1,2}$ being a different symbol from $a_{2,1}$, a_{34} different from a_{43}, and so on.

In the case of a general determinant like

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots\ldots a_{2n} \\ a_{31} & a_{32} & a_{33} \ldots\ldots a_{3n} \\ \ldots & \ldots & \ldots\ldots\ldots \\ a_{n1} & a_{n2} & a_{n3} \ldots\ldots a_{nn} \end{vmatrix}$$

where such elements are used, and where the first suffix corresponds with the row to which the element belongs and the second suffix with the column, we can specify by a symbol any element which is omitted from the determinant as written, and assign to its proper position any such element whose symbol is mentioned, with greater ease than is possible in the case of determinants with single-suffix elements. Besides, as only one letter is employed, we may, if we choose, denote such a determinant itself by a shorter symbol even than

$$D(a_{11}\,a_{22}\,a_{33}\ldots a_{nn}) \text{ or } |a_{11}\,a_{22}\,a_{33}\ldots a_{nn}|,$$

viz. by

$$D(a_{1n}) \text{ or } |a_{1n}|.$$

§ 57. When r rows of a determinant of the n^{th} order are deleted, the number of elements deleted is nr, and when subsequently n columns are deleted, the number of new elements thus struck out is $nr - r^2$; so that the number of elements left is

$$n^2 - 2nr + r^2,$$

$$\text{i.e.} \qquad (n-r)^2.$$

If any number of the rows of a determinant be deleted and as many of the columns, the determinant whose elements are in order the elements thus left is called a MINOR of the original determinant. Thus, taking

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \text{ and } \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix},$$

and deleting, in the former the second and third rows and second and fourth columns, and in the latter the second and third and fourth rows and first and third and fourth columns, we see that

$$\begin{vmatrix} a_{11} & a_{13} & a_{15} \\ a_{41} & a_{43} & a_{45} \\ a_{51} & a_{53} & a_{55} \end{vmatrix} \text{ and } \begin{vmatrix} a_{12} & a_{15} \\ a_{52} & a_{55} \end{vmatrix}$$

are minors of $|a_{11}a_{22}a_{33}a_{44}a_{55}|$.

The minors obtained by the deletion of one row and one column are called the *principal minors* of the determinant. Thus $|a_{21}a_{32}a_{43}a_{54}|$, $|a_{12}a_{23}a_{34}a_{45}|$, $|a_{11}a_{22}a_{44}a_{55}|$, ... are principal minors of $|a_{11}a_{22}a_{33}a_{44}a_{55}|$.

Two minors which are such that the rows and columns deleted to obtain the one are exactly those not deleted in obtaining the other are called *complementary minors*. Thus the complementary minors of the two minors first given above are

$$\begin{vmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{vmatrix} \text{ and } \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix},$$

the elements of which, it may be noticed, are the elements that are common to the deleted rows and the deleted columns; also, $|a_{22}a_{33}|$ and $|a_{11}a_{44}a_{55}|$ are complementary minors, and $|a_{22}a_{33}a_{44}a_{55}|$ is the complementary minor of a_{11}.

If the original determinant be of the n^{th} degree, and the number of deleted rows and of deleted columns be r, the resulting minor is of the $(n-r)^{\text{th}}$ degree, and its complementary of the r^{th}.

§ 58. The minor of $D(a_{1n})$ obtained by deleting the h^{th} row and r^{th} column may be shortly denoted by

$$D^{(h}_{(r},$$

that obtained by deleting the h^{th} and k^{th} rows and r^{th} and s^{th} columns by

$$D^{(h,k}_{(r,s},$$

and so on.

§ 59. Principal minors have already come into consideration in the theorems of §§ 44, 46, the former of which, in the notation just given, takes the form—

$$\text{the co-factor of } a_{hr} \text{ in } D(a_{1n}) = (-1)^{h+r} D^{(h}_{(r}. \qquad \ldots(1)$$

Likewise, it having been shown in § 50 that

$$\text{the co-factor of } a_{hr} \text{ in } D(a_{1n}) = \frac{\partial D}{\partial a_{hr}},$$

there thus has been implied another fact regarding them, viz.

$$D^{(h}_{(r} = (-1)^{h+r} \frac{\partial D}{\partial a_{hr}}. \qquad \ldots(2)$$

It will now appear that these are the simplest cases of more general theorems.

§ 60. *If a determinant be differentiated with respect to the element common to the first row and column deleted to form any minor, this differential coefficient be differentiated with respect to the element common to the next row and column deleted, and so on, the result equals the said minor or differs from it only in sign, according as the sum of the numbers indicating the deleted rows and columns is even or odd.*

Let $D(a_{1n})$ be the determinant, and let the rows which are deleted to form the minor be in order the h^{th}, k^{th}, l^{th}, ... and the columns the r^{th}, s^{th}, u^{th}, ..., so that the minor would be denoted by $D^{(h,k,l,\ldots}_{(r,s,u,\ldots}$. Then (§ 59) we have

$$D^{(h}_{(r} = (-1)^{h+r} \frac{\partial D}{\partial a_{hr}};$$

and, since the element of the k^{th} row and s^{th} column of D is the element of the $(k-1)^{\text{th}}$ row and $(s-1)^{\text{th}}$ column of $D^{(h}_{(r}$, we have further (§ 59)

$$D^{(h,k}_{(r,s} = (-1)^{k-1+s-1} \frac{\partial}{\partial a_{ks}} \left\{ (-1)^{h+r} \frac{\partial D}{\partial a_{hr}} \right\},$$

$$= (-1)^{k+s+h+r} \frac{\partial^2 D}{\partial a_{ks} \partial a_{hr}};$$

so that if the minor be of the m^{th} degree, we have generally

$$D^{(h,k,l,\ldots}_{(r,s,u,\ldots} = (-1)^{h+k+l+\ldots+r+s+u+\ldots} \frac{\partial^m D}{\partial a_{hr} \partial a_{ks} \partial a_{lu} \ldots .}.$$

It has to be carefully noted that in the identity here proved $h < k < l < \ldots .$ and $r < s < u < \ldots .$ If this were not the case, there would still be equality as to magnitude, but the sign would be otherwise determined than from the sum of $h, k, l, \ldots, r, s, u, \ldots .$

§ 61. *In a determinant the co-factor of the product of any number of elements which may come together in one term is equal to the result of differentiating the determinant in succession with respect to the said elements.*

Just as by differentiating $D(a_{1n})$, or D say, with respect to a_{hr} we obtain the co-factor of a_{hr} in D, so by differentiating the result with respect to an element of another row and column, a_{ks} say, we obtain the co-factor of a_{ks} in $\frac{\partial D}{\partial a_{hr}}$; and thus we see that

$$\text{the co-factor of } a_{hr} a_{ks} \text{ in } D = \frac{\partial^2 D}{\partial a_{hr} \partial a_{ks}}.$$

Similarly,

$$\text{the co-factor of } a_{hr} a_{ks} a_{lu} \text{ in } D = \frac{\partial^3 D}{\partial a_{hr} \partial a_{ks} \partial a_{lu}},$$

and so generally.

Example. We know (§ 46) that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \quad \text{or} \quad \Delta = a_{11}\Delta_{(1}^{(1} - a_{21}\Delta_{(1}^{(2} + a_{31}\Delta_{(1}^{(3} - a_{41}\Delta_{(1}^{(4} + a_{51}\Delta_{(1}^{(5}\ ;$$

$$\therefore \text{ co-factor of } a_{31} = \frac{\partial \Delta}{\partial a_{31}} = \Delta_{(1}^{(3},$$

$$= \begin{vmatrix} a_{12} & a_{13} & a_{14} & a_{15} \\ a_{22} & a_{23} & a_{24} & a_{25} \\ a_{42} & a_{43} & a_{44} & a_{45} \\ a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix},$$

$$= -a_{15}|a_{22}a_{43}a_{54}| + a_{25}|a_{12}a_{43}a_{54}| - a_{45}|a_{12}a_{23}a_{54}| + a_{55}|a_{12}a_{23}a_{44}|;$$

$$\therefore \text{ co-factor of } a_{31}a_{45} = \frac{\partial^2 \Delta}{\partial a_{31}\,\partial a_{45}} = -|a_{12}a_{23}a_{54}|,$$

$$= (-1)^{5+4+3+1}\Delta_{(1,5}^{(3,4}.$$

The effect of the first differentiation here is, as it were, to delete the third row and first column of Δ, the effect of the second differentiation to delete the fourth row and fifth column. But the same final result would be produced upon Δ by deleting first its third row and fifth column and then its fourth row and first column. Differentiating then with respect to a_{35} and afterwards with respect to a_{41}, we find

$$\text{the co-factor of } a_{41}a_{35} = \frac{\partial^2 \Delta}{\partial a_{41}\,\partial a_{35}} = |a_{12}a_{23}a_{54}|.$$

Hence the minor $|a_{12}a_{23}a_{54}|$ is the co-factor of both $-a_{31}a_{45}$ and $a_{41}a_{35}$ in Δ; therefore it is the co-factor of $-(a_{31}a_{45} - a_{41}a_{35})$ or $-|a_{31}a_{45}|$; in other words, the co-factor (with sign changed) of the minor $|a_{31}a_{45}|$ is its complementary minor,—an instance of a theorem soon to be given.

§ 62. *Any row and column of a determinant being selected, if the element common to them be multiplied by its co-factor in the determinant, and every product of another element of the row by another element of the column be multiplied by its co-factor, the sum of the results is equal to the given determinant.*

Let the selected row and column be

$$\begin{array}{ccccccccc}
 & & & & a & & & & \\
 & & & & b & & & & \\
 & & & & c & & & & \\
 & & & & \vdots & & & & \\
 & & & & g & & & & \\
m & n & \ldots\ldots & r & o & p & \ldots\ldots & x. \\
 & & & & h & & & & \\
 & & & & \vdots & & & & \\
 & & & & t & & & &
\end{array}$$

The multiplication of o by its co-factor gives all the terms of the determinant which contain o. As each of the terms containing r must also contain one of the elements of the column and cannot contain o, we see that by multiplying $ra, rb, rc, \ldots, rg, rh, \ldots, rt$ by their respective co-factors, we obtain all the terms containing r. Similarly by multiplying $pa, pb, pc, \ldots, pg, ph, \ldots, pt$ by their respective co-factors, we obtain all the terms containing p. Consequently if we continue this process we shall finally have every term in which one of the elements of the row occurs, that is to say, we shall have the full development of the determinant,—and this was the theorem to be proved.

EXAMPLE:—

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{22} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} - a_{21}a_{12} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} + a_{21}a_{32} \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix} - a_{21}a_{42} \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix}$$

$$+ a_{23}a_{12} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} - a_{23}a_{32} \begin{vmatrix} a_{11} & a_{14} \\ a_{41} & a_{44} \end{vmatrix} + a_{23}a_{42} \begin{vmatrix} a_{11} & a_{14} \\ a_{31} & a_{34} \end{vmatrix}$$

$$- a_{24}a_{12} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + a_{24}a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{41} & a_{43} \end{vmatrix} - a_{24}a_{42} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}.$$

§ 63. *Any determinant of the* nth *order may be developed in a series of terms, the first of which is got from the given determinant by changing all the elements of the principal diagonal into zero, the next* n *by multiplying each element of the principal diagonal by its co-factor in the determinant and altering the said co-factors as the given determinant was altered, the next* $\frac{1}{2}$n(n−1) *by multiplying the product of each pair of elements of the principal diagonal by its co-factor altered as before, and so on, the last term being simply the product of the elements of the principal diagonal.*

By taking the given determinant and changing all the elements of the principal diagonal into zero, we delete all the terms containing any of these elements; also no other terms are thereby deleted. Thus the altered determinant represents the sum of the terms which are independent of the elements in question.

By multiplying an element of the principal diagonal by its co-factor we obtain exactly all the terms containing that element. This co-factor, however, has its principal diagonal composed of elements from the original principal diagonal. If we therefore change these into zero, the altered product will represent the sum of the terms which contain the element in question and none of its fellows in the principal diagonal.

In this way it is seen that the expansion specified in the theorem gives, first, all the terms of the determinant involving *no* element of the principal diagonal; secondly, all those involving only *one* element; thirdly, all those involving only two elements, and so on: so that the full number of terms is in the end obtained.

It is worthy of notice that there is a break in the series:

we pass from those terms involving only $n-2$ elements of the diagonal to the term involving them all, there being no term involving only $n-1$ of them.

EXAMPLE:—

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{vmatrix} + a_{11}\begin{vmatrix} 0 & a_{23} \\ a_{32} & 0 \end{vmatrix} + a_{22}\begin{vmatrix} 0 & a_{13} \\ a_{31} & 0 \end{vmatrix} + a_{33}\begin{vmatrix} 0 & a_{12} \\ a_{21} & 0 \end{vmatrix} + a_{11}a_{22}a_{33}.$$

§ 64. *In a determinant of the* n*th order the full number of terms which are independent of the elements of the principal diagonal is*

$$1.2.3\ldots n\left\{\frac{1}{1.2}-\frac{1}{1.2.3}+\frac{1}{1.2.3.4}-\ldots\ldots+(-1)^n\frac{1}{1.2.3\ldots n}\right\}.$$

Let $|a_{11}a_{12}\ldots a_{rr}\ldots a_{nn}|$ be the determinant, and $\phi(n)$, as yet unknown, the number of terms in it of the kind referred to; so that $\phi(3)=2$ and $\phi(2)=1$. Then, as we have seen (§ 63), $\phi(n)$ equals the number of terms in

$$\begin{vmatrix} 0 & a_{12} & a_{13} & \ldots\ldots a_{1,r-1} & a_{1r} & a_{1,r+1} & \ldots\ldots a_{1n} \\ a_{21} & 0 & a_{23} & \ldots\ldots a_{2,r-1} & a_{2r} & a_{2,r+1} & \ldots\ldots a_{2n} \\ a_{31} & a_{32} & 0 & \ldots\ldots a_{3,r-1} & a_{3r} & a_{3,r+1} & \ldots\ldots a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{r-1,1} & a_{r-1,2} & a_{r-1,3} & \ldots\ldots 0 & a_{r-1,r} & a_{r-1,r+1} & \ldots\ldots a_{r-1,n} \\ a_{r,1} & a_{r,2} & a_{r,3} & \ldots\ldots a_{r,r-1} & 0 & a_{r,r+1} & \ldots\ldots a_{r,n} \\ a_{r+1,1} & a_{r+1,2} & a_{r+1,3} & \ldots\ldots a_{r+1,r-1} & a_{r+1,r} & 0 & \ldots\ldots a_{r+1,n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots\ldots a_{n,r-1} & a_{nr} & a_{n,r+1} & \ldots\ldots 0 \end{vmatrix} \quad \ldots(1)$$

Now this determinant is equal (§ 46) to the aggregate of $n-1$ items such as

$$(-1)^{1+r}a_{1r}\begin{vmatrix} a_{21} & 0 & a_{23} & \cdots\cdots a_{2,r-1} & a_{2,r+1} & \cdots\cdots a_{2n} \\ a_{31} & a_{32} & 0 & \cdots\cdots a_{3,r-1} & a_{3,r+1} & \cdots\cdots a_{3n} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots\cdots \\ a_{r-1,1} & a_{r-1,2} & a_{r-1,3} & \cdots\cdots 0 & a_{r-1,r+1} & \cdots\cdots a_{r-1,n} \\ a_{r,1} & a_{r,2} & a_{r,3} & \cdots\cdots a_{r,r-1} & a_{r,r+1} & \cdots\cdots a_{r,n} \\ a_{r+1,1} & a_{r+1,2} & a_{r+1,3} & \cdots\cdots a_{r+1,r-1} & 0 & \cdots\cdots a_{r+1,n} \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots\cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots\cdots a_{n,r-1} & a_{n,r+1} & \cdots\cdots 0 \end{vmatrix}.$$

Transferring the $(r-1)^{\text{th}}$ row to the top, and replacing the resulting determinant by two determinants (§ 28) differing from it only in having for their first rows

$$\begin{array}{lllllll} & 0 & a_{r2}\ a_{r3}\ \ldots\ldots & a_{r,r-1} & a_{r,r+1} & \ldots\ldots & a_{rn}, \\ \text{and} & a_{r1} & 0\ \ 0\ \ \ldots\ldots & 0 & 0 & \ldots\ldots & 0, \end{array}$$

respectively, we find that the first of the two is a determinant like (1), but of the $(n-1)^{\text{th}}$ order, and therefore having $\phi(n-1)$ terms; and that the second is expressible (§ 46) as the product of a_{r1} and a similar determinant of the $(n-2)^{\text{th}}$ order. Hence

$$\phi(n) = (n-1)\left\{\phi(n-1)+\phi(n-2)\right\},$$

$$\text{and } \therefore\ \phi(n) - n\phi(n-1) = -\left\{\phi(n-1)-(n-1)\phi(n-2)\right\};$$

that is to say, $\phi(n)-n\phi(n-1)$ remains the same in magnitude for two consecutive values, and therefore for all values, of n. But

$$\text{when } n=3,\ \phi(n)-n\phi(n-1) = \phi(3)-3\phi(2) = 2-3 = -1,$$

$$\therefore\ \text{when } n=4,\ \phi(n)-n\phi(n-1) = +1,$$

$$\text{and, generally, } \phi(n)-n\phi(n-1) = (-1)^n,$$

$$\text{or} \qquad \phi(n) = (-1)^n + n\phi(n-1).$$

Beginning with this, and making repeated use of it, we have

$$
\begin{aligned}
&\phi(n)\\
&\quad=(-1)^n+n\left\{(-1)^{n-1}+(n-1)\phi(n-2)\right\},\\
&\quad=(-1)^n+(-1)^{n-1}n+n(n-1)\left\{(-1)^{n-2}+(n-2)\phi(n-3)\right\},\\
&\quad=(-1)^n+(-1)^{n-1}n+(-1)^{n-2}n(n-1)\\
&\qquad\qquad\qquad+n(n-1)(n-2)\left\{(-1)^{n-3}+(n-3)\phi(n-4)\right\},\\
&\quad=(-1)^n+(-1)^{n-1}n+(-1)^{n-2}n(n-1)\\
&\qquad\qquad\qquad+(-1)^{n-3}n(n-1)(n-2)+\ldots\ldots\ldots\ldots\ldots\\
&\qquad\qquad\qquad+n(n-1)(n-2)\ldots5.4\left\{(-1)^3+3\phi(2)\right\},
\end{aligned}
$$

whence, if we substitute for $\phi(2)$ its value 1, and reverse the order of the terms, we have the result desired.

A determinant of the n^{th} order, having only n zero-elements, and these occupying the principal diagonal, has been called by Sylvester an *invertebrate* or *zero-axial* determinant.

EXAMPLE. Find the number of possible arrangements of n things a_1, a_2, a_3, ..., a_n, subject to the condition that no one shall be in its original place.

We may have a_1 in any place except the first, a_2 in any place except the second, and so on—data which we may present to the eye in the form

$$
\begin{array}{lllll}
(\;) & a_1 & a_1 & a_1\ldots\ldots & a_1\\
a_2 & (\;) & a_2 & a_2\ldots\ldots & a_2\\
a_3 & a_3 & (\;) & a_3\ldots\ldots & a_3\\
a_4 & a_4 & a_4 & (\;)\ldots\ldots & a_4\\
\multicolumn{5}{l}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}\\
a_n & a_n & a_n & a_n\ldots\ldots & (\;),
\end{array}
$$

an a_1 being written in the places which it is allowable for a_1 to occupy, and ( ) signifying that the suffixed letter found in the same line with it may not occupy its place. Looking to this table we see that for the first place in any of the arrangements we may take any letter that is in the first column; for the second place any letter that is in the second column, provided it be not in the same line with the letter taken from the first column; for the third place any letter that is in the third column, provided it be not in the same line with either of the letters previously taken, and so on. This law of formation, however, is identical with that in accordance with which the terms of a determinant are got from the elements: so that the problem we are concerned with is transformed into finding the full number of terms of a zero-axial determinant of the n^{th} order. Hence (§ 64) the required number of arrangements is

$$1.2.3\ldots n\left\{\frac{1}{1.2}-\frac{1}{1.2.3}+\frac{1}{1.2.3.4}-\ldots.+(-1)^n\frac{1}{1.2.3\ldots n}\right\}.$$

EXERCISES. SET XI.

Write down the complementaries of the following minors of $|a_0b_1c_2d_3e_4f_5|$:—

1. b_3. 2. $|c_2e_4|$. 3. $|c_0f_5|$. 4. $|b_0c_5|$.
5. $|d_2e_3f_4|$. 6. $|b_1c_3d_4|$. 7. $|b_0e_2f_4|$. 8. $|c_0d_4e_2|$.

9. What are the complementary minors of $|a_{13}a_{35}|$ and $|a_{12}a_{35}a_{56}|$ in $|a_{01}a_{12}a_{23}a_{34}a_{45}a_{56}|$?

10. Express as determinants

$$\frac{\partial^2}{\partial a_{25}\partial a_{53}}|a_{16}| \text{ and } \frac{\partial^3}{\partial a_{13}\partial a_{42}\partial a_{35}}|a_{01}a_{12}a_{23}a_{34}a_{45}a_{56}|.$$

11. What is the co-factor of $a_{22}a_{43}$ in $|a_{21}a_{32}a_{43}a_{54}a_{65}|$ and in $|a_{01}a_{12}a_{23}a_{34}a_{45}a_{56}|$?

12. Express in symbols the theorem regarding the effect of transposing two rows of a determinant.

13. Find the single number which is the equivalent of

$$\begin{vmatrix} 1 & a & b & c \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{vmatrix}.$$

14. Find the final expansion of

$$\begin{vmatrix} 0 & -b & -c & -d \\ b & 0 & -e & -f \\ c & e & 0 & -g \\ d & f & g & 0 \end{vmatrix}.$$

15. Express in terms arranged according to descending powers of x

$$\begin{vmatrix} x & -1 & -1 & -1 \\ 1 & x^3 & -1 & -1 \\ 1 & 1 & x^5 & -1 \\ 1 & 1 & 1 & x^7 \end{vmatrix}.$$

16. Find the quadratic in x, y, z equivalent to

$$\begin{vmatrix} 0 & x & y & z \\ x & a & h & g \\ y & h & b & f \\ z & g & f & c \end{vmatrix}.$$

17. Find the quadratic in x, y, z, w equivalent to

$$\begin{vmatrix} 0 & x & y & z & w \\ x & 1 & -1 & -1 & -1 \\ y & -1 & 1 & -1 & -1 \\ z & -1 & -1 & 1 & -1 \\ w & -1 & -1 & -1 & 1 \end{vmatrix}.$$

18. If $\phi(n)$ denote as in § 64 the number of terms of a zero-axial determinant of the n^{th} order, prove with the help of § 63 that

$$\phi(n) + \frac{n}{1}\phi(n-1) + \frac{n(n-1)}{1.2}\phi(n-2) + \ldots + \frac{n(n-1)}{1.2}\phi(2) + n\phi(1) + 1 = 1.2.3 \ldots n.$$

19. Find the quadratic in x, y, z equivalent to

$$\begin{vmatrix} 1 & 0 & 0 & ax+hy+gz \\ 0 & 1 & 0 & hx+by+fz \\ 0 & 0 & 1 & gx+fy+cz \\ x & y & z & 0 \end{vmatrix}.$$

20. Find the number of minors of the $(n-2)^{\text{th}}$ order in a determinant of the n^{th} order.

21. Prove that

$$\begin{vmatrix} 1 & 0 & 0 & 0 & ax+hy+gz \\ 0 & 1 & 0 & 0 & hx+by+fz \\ 0 & 0 & 1 & 0 & gx+fy+cz \\ 0 & 0 & 0 & 1 & lx+my+nz \\ x & y & z & 1 & k \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & ax+hy+gz+l \\ 0 & 1 & 0 & hx+by+fz+m \\ 0 & 0 & 1 & gx+fy+cz+n \\ x & y & z & k \end{vmatrix},$$

$$= -(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + lx + my + nz - k).$$

22. k particular elements of the principal diagonal of a determinant of the n^{th} order being fixed upon, find the number of terms which contain these elements and no other elements of the same diagonal.

23. Find the difference between the number of positive and the number of negative terms in a zero-axial determinant of the n^{th} order.

24. Prove that

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1+a & 1+b & 1+c \\ 1 & a+1 & 0 & a+b & a+c \\ 1 & b+1 & b+a & 0 & b+c \\ 1 & c+1 & c+a & c+b & 0 \end{vmatrix} = 2^3 \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}.$$

25. Prove that

$$\begin{vmatrix} a_1a_2 & a_1 & a_2 & 1 \\ b_1b_2 & b_2 & b_1 & 1 \\ c_1c_2 & c_1 & c_2 & 1 \\ d_1d_2 & d_2 & d_1 & 1 \end{vmatrix} = \left\{ \begin{array}{l} (a_1-b_2)(b_1-c_2)(c_1-d_2)(d_1-a_2) \\ -(a_2-b_1)(b_2-c_1)(c_2-d_1)(d_2-a_1). \end{array} \right.$$

26. From the theorem of Exercise 18 above, prove that

$$\phi(n) = \left| \begin{array}{cccccc} n! & C_{n,1} & C_{n,2} & \ldots\ldots & C_{n,n-1} & C_{n,n} \\ (n-1)! & 1 & C_{n-1,1} & \ldots\ldots & C_{n-1,n-2} & C_{n-1,n-1} \\ (n-2)! & 0 & 1 & \ldots\ldots & C_{n-2,n-3} & C_{n-2,n-2} \\ \multicolumn{6}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ 1! & 0 & 0 & \ldots\ldots & 1 & C_{1,1} \\ 1 & 0 & 0 & \ldots\ldots & 0 & 1 \end{array} \right|,$$

where $n!$ stands for $1.2.3 \ldots . n$, and $C_{n,r}$ for $\dfrac{n(n-1)(n-2)\ldots .(n-r+1)}{1.2.3\ldots . r}$.

§ 65. *The product of a determinant of the* n*th* *degree by an expression of* n *terms is equal to the sum of* n *determinants, the first of which is got from the given determinant by multiplying each element of the first row by the corresponding term of the given expression, the second by multiplying similarly each element of the second row, the third by multiplying similarly each element of the third row, and* so *on.*

Let the given determinant be

$$D(a_{11}\,a_{22}\,a_{33}\,\ldots.\,a_{nn}) \quad \text{or} \quad D,$$

and the given expression

$$\mu_1 + \mu_2 + \mu_3 + \ldots + \mu_n,$$

then the n determinants referred to are

$$\begin{vmatrix} \mu_1 a_{11} & \mu_2 a_{12} \ldots. & \mu_n a_{1n} \\ a_{21} & a_{22} \ldots. & a_{2n} \\ \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} \ldots. & a_{nn} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} \ldots. & a_{1n} \\ \mu_1 a_{21} & \mu_2 a_{22} \ldots. & \mu_n a_{2n} \\ \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} \ldots. & a_{nn} \end{vmatrix}, \ldots\ldots\ldots\ldots$$

Now the coefficient of μ_1 in the first of them is evidently $a_{11}A_{11}$, in the second $a_{21}A_{21}$, in the third $a_{21}A_{31}$, and so on: therefore in the sum of the n determinants the coefficient of μ_1 is

$$a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + \ldots. + a_{n1}A_{n1} \quad \text{or} \quad D.$$

Similarly the coefficient of μ_2 is seen to be

$$a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} + \ldots. + a_{n2}A_{n2} \quad \text{or} \quad D,$$

and so on. Hence the sum of the n determinants is

$$(\mu_1 + \mu_2 + \mu_3 + \ldots. + \mu_n)D.$$

§ 66. We have already had (Ex. 12, Set II.) an instance of the product of two determinants being itself expressed as a determinant, viz. the case in which all the determinants are of the second order. Let us now consider the corresponding example for the case of determinants of the third order.

Taking the determinants[*]

$$\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & K \end{vmatrix} \quad \text{or} \quad \Delta, \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} \quad \text{or} \quad \Delta',$$

and

$$\begin{vmatrix} aA + bB + cC & dA + eB + fC & gA + hB + kC \\ aD + bE + cF & dD + eE + fF & gD + hE + kF \\ aG + bH + cK & dG + eH + fK & gG + hH + kK \end{vmatrix} \quad \text{or} \quad \Delta'',$$

and comparing them, we see that the first row of Δ'' is formed from the first row of Δ and all the rows of Δ', the second row of Δ'' from the second row of Δ and all the rows of Δ', the third row of Δ'' from the third row of Δ and all the rows of Δ'. Looking closer, we observe that the element of the first row and first column of Δ'', viz. $aA+bB+cC$, is formed by multiplying each element of the first row of Δ by the corresponding element of the first row of Δ', and adding the products thus formed, that the element of the first row and second column of Δ'', viz. $dA+eB+fC$, is formed in like manner from the first row of Δ and the second row of Δ'; and that, generally, the element of the p^{th} row and q^{th} column of Δ'' is formed in this manner from the p^{th} row of Δ and the q^{th} row of Δ'.

Now the elements of Δ'' being all trinomial, the determinant may be partitioned (§ 29) into twenty-seven determinants having all their elements monomial. Of these, however, twenty-one will be found to vanish on account of the existence in them of identical columns; thus,

$$\begin{vmatrix} aA & dA & kC \\ aD & dD & kF \\ aG & dG & kK \end{vmatrix} = adk \begin{vmatrix} A & A & C \\ D & D & F \\ G & G & K \end{vmatrix} = 0.$$

---

[*] In using capital letters here no reference is intended to the notation of § 49.

Indeed it is clear that in forming the twenty-seven determinants we need not take the set of first terms in the first column of Δ'' along with the set of first terms in either of the other two columns, nor the set of second terms in the first column along with the set of second terms in either of the other two columns, nor the set of third terms in the first column along with the set of third terms in either of the other two columns. The only determinants which do not vanish will therefore be composed of a set of first terms taken from one column, a set of second terms taken from another column, and a set of third terms taken from the remaining column; and the number of them will consequently be the number of permutations of the numbers 1, 2, 3, that is 6. In agreement with this the result will be found to be

$$\Delta''=\begin{vmatrix} aA & eB & kC \\ aD & eE & kF \\ aG & eH & kK \end{vmatrix}+\begin{vmatrix} aA & fC & hB \\ aD & fF & hE \\ aG & fK & hH \end{vmatrix}+\begin{vmatrix} bB & dA & kC \\ bE & dD & kF \\ bH & dG & kK \end{vmatrix}$$

$$+\begin{vmatrix} bB & fC & gA \\ bE & fF & gD \\ bH & fK & gG \end{vmatrix}+\begin{vmatrix} cC & dA & hB \\ cF & dD & hE \\ cK & dG & hH \end{vmatrix}+\begin{vmatrix} cC & eB & gA \\ cF & eE & gD \\ cK & eH & gG \end{vmatrix},$$

$$=aek\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & K \end{vmatrix}+afh\begin{vmatrix} A & C & B \\ D & F & E \\ G & K & H \end{vmatrix}+bdk\begin{vmatrix} B & A & C \\ E & D & F \\ H & G & K \end{vmatrix}$$

$$+bfg\begin{vmatrix} B & C & A \\ E & F & D \\ H & K & G \end{vmatrix}+cdh\begin{vmatrix} C & A & B \\ F & D & E \\ K & G & H \end{vmatrix}+ceg\begin{vmatrix} C & B & A \\ F & E & D \\ K & H & G \end{vmatrix}$$

$$=\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & K \end{vmatrix}(aek-afh-bdk+bfg+cdh-ceg),$$

$$= \begin{vmatrix} A & B & C \\ D & E & F \\ G & H & K \end{vmatrix} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}.$$

§ 67. *If two determinants Δ, Δ' of the same order be given, and a new determinant Δ'' be formed such that in every case the element of its* p^{th} *row and* q^{th} *column is obtained by multiplying each element of the* p^{th} *row of* Δ' *by the corresponding element of the* q^{th} *row of* Δ, *and adding the products thus found, then* $\Delta'' = \Delta\Delta'$.

Let the two determinants Δ, Δ' be

$$\begin{vmatrix} a_1 & a_2 & a_3 \dots a_n \\ b_1 & b_2 & b_3 \dots b_n \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ l_1 & l_2 & l_3 \dots l_n \end{vmatrix} \text{ and } \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \dots \alpha_n \\ \beta_1 & \beta_2 & \beta_3 \dots \beta_n \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \lambda_1 & \lambda_2 & \lambda_3 \dots \lambda_n \end{vmatrix}$$

respectively.

Observing the first row of Δ and the first row of Δ', we see that the element of the first row and first column of Δ'' will be

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \dots + a_n\alpha_n.$$

Again, taking the second row of Δ and as before the first row of Δ', we obtain the element of the second row and first column of Δ'', viz.

$$b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 + \dots + b_n\alpha_n.$$

Similarly we find the element of the third row and first column of Δ'' to be

$$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + \dots + c_n\alpha_n;$$

and so on. The first column of Δ'' is thus

$$\begin{array}{l} a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n, \\ b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 + \ldots + b_n\alpha_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ l_1\alpha_1 + l_2\alpha_2 + l_3\alpha_3 + \ldots + l_n\alpha_n, \end{array}$$

and is therefore so constituted that if the plus signs and factors taken from Δ be deleted we have the first line of Δ' repeated n times, and if the plus signs and factors taken from Δ' be deleted we have in order all the lines of Δ. The second column differs from this only in having β in place of α, the third in having γ, and so on; so that Δ'' is

$$\left|\begin{array}{llll} a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n, & a_1\beta_1 + \ldots + a_n\beta_n, & \ldots\ldots, & a_1\lambda_1 + \ldots + a_n\lambda_n, \\ b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n, & b_1\beta_1 + \ldots + b_n\beta_n, & \ldots\ldots, & b_1\lambda_1 + \ldots + b_n\lambda_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ l_1\alpha_1 + l_2\alpha_2 + \ldots + l_n\alpha_n, & l_1\beta_1 + \ldots + l_n\beta_n, & \ldots\ldots, & l_1\lambda_1 + \ldots + l_n\lambda_n, \end{array}\right|$$

Now this we know (§ 29) can be partitioned into n^n determinants, the columns of each of which are got by taking from each of the columns of Δ'' a set of terms in the same vertical line. Of these determinants, however, those containing two columns taken from corresponding places in the columns of Δ'' may be neglected, since, when the common factor of the elements of each column of such a determinant is separated from them, the determinant must have two columns identical, and therefore be equal to zero. Forming a specimen of the remaining determinants, which are thus seen to be $n!$ in number, we choose from the first column of Δ'' the set of second terms say, from the second column the set of fifth terms, from the third column the set of fourth terms, from the n^{th} column the set of first terms, and similarly a set from each of the other columns, the result being the determinant

$$\begin{vmatrix} a_2\alpha_2 & a_5\beta_5 & a_4\gamma_4 & \dots & a_1\lambda_1 \\ b_2\alpha_2 & b_5\beta_5 & b_4\gamma_4 & \dots & b_1\lambda_1 \\ c_2\alpha_2 & c_5\beta_5 & c_4\gamma_4 & \dots & c_1\lambda_1 \\ \dots & \dots & \dots & \dots & \dots \\ l_2\alpha_2 & l_5\beta_5 & l_4\gamma_4 & \dots & l_1\lambda_1 \end{vmatrix}.$$

Now this is equal to

$$\alpha_2\beta_5\gamma_4\dots\lambda_1 \begin{vmatrix} a_2 & a_5 & a_4 & \dots & a_1 \\ b_2 & b_5 & b_4 & \dots & b_1 \\ c_2 & c_5 & c_4 & \dots & c_1 \\ \dots & \dots & \dots & \dots & \dots \\ l_2 & l_5 & l_4 & \dots & l_1 \end{vmatrix},$$

of which, laying aside the question of sign, the first factor $\alpha_2\beta_5\gamma_4\dots\lambda_1$ is a term of Δ' and the other factor is Δ. Further, since the number of inversions, say p, in the suffixes of the first factor is the same as in the suffixes of each row of the second factor, and since every transposition made with a pair of rows in the second factor in order to get rid of an inversion alters the sign of the determinant, if we wish to have *no* inversions, that is to say, if we wish to substitute Δ for the determinant factor in question, we must prefix to $\alpha_2\beta_5\gamma_4\dots\lambda_1$ the sign $(-1)^p$, in other words, its own proper sign as a term of Δ'. The $n!$ determinants whose sum is equal to Δ'' are thus seen to be each expressible as the product of two factors, one factor being Δ in every case, and the other being in succession all the terms of Δ'. Hence

$$\begin{aligned}\Delta'' &= \Delta \times \Sigma(\pm a_1\beta_2\gamma_3 \cdots \lambda_n)\\ &= \Delta \times \Delta'.\end{aligned}$$

Example:—

$$\begin{vmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 1 & 4 \\ 0 & 2 & 1 & -1 \\ 2 & 3 & 0 & -4 \end{vmatrix} \times \begin{vmatrix} -1 & 4 & 2 & -1 \\ 2 & -1 & 3 & -2 \\ 0 & 2 & -1 & 1 \\ 3 & 0 & 4 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1+8+2-1, & 2-2+3-2, & 0+4-1+1, & 3+0+4-1 \\ -3+0+2-4, & 6-0+3-8, & 0+0-1+4, & 9+0+4-4 \\ 0+8+2+1, & 0-2+3+2, & 0+4-1-1, & 0+0+4+1 \\ -2+12+0+4, & 4-3+0+8, & 0+6-0-4, & 6+0+0+4 \end{vmatrix},$$

$$= \begin{vmatrix} 8 & 1 & 4 & 6 \\ -5 & 1 & 3 & 9 \\ 11 & 3 & 2 & 5 \\ 14 & 9 & 2 & 10 \end{vmatrix}.$$

§ 68. Since it is possible to alter, in accordance with previously established theorems, the form of either or both of the two given determinants before going through the process of forming the determinant which is to be their product, it is clear that in this way there may be obtained more than one form of result. Thus, using the theorem (§ 24) regarding the transformation of rows into columns, we have

$$\begin{aligned}
\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix} &= \begin{vmatrix} b_1\beta_1 + c_1\gamma_1 & b_1\beta_2 + c_1\gamma_2 \\ b_2\beta_1 + c_2\gamma_1 & b_2\beta_2 + c_2\gamma_2 \end{vmatrix},\\
= \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \begin{vmatrix} \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{vmatrix} &= \begin{vmatrix} b_1\beta_1 + c_1\beta_2 & b_1\gamma_1 + c_1\gamma_2 \\ b_2\beta_1 + c_2\beta_2 & b_2\gamma_1 + c_2\gamma_2 \end{vmatrix},\\
= \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix} &= \begin{vmatrix} b_1\beta_1 + b_2\gamma_1 & b_1\beta_2 + b_2\gamma_2 \\ c_1\beta_1 + c_2\gamma_1 & c_1\beta_2 + c_2\gamma_2 \end{vmatrix},\\
= \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \begin{vmatrix} \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{vmatrix} &= \begin{vmatrix} b_1\beta_1 + b_2\beta_2 & b_1\gamma_1 + b_2\gamma_2 \\ c_1\beta_1 + c_2\beta_2 & c_1\gamma_1 + c_2\gamma_2 \end{vmatrix}.
\end{aligned}$$

§ 69. The form of the result in the case in which the two determinants to be multiplied are identical is worthy of notice, each of the elements situated on one side of the principal diagonal being identical with that similarly situated on the other side. Thus, taking $|\alpha_1\beta_2\gamma_3|^2$ as an instance, we have

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}^2$$

$$= \begin{vmatrix} \alpha_1^2 + \beta_1^2 + \gamma_1^2, & \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2, & \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3 \\ \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2, & \alpha_2^2 + \beta_2^2 + \gamma_2^2, & \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 \\ \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3, & \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3, & \alpha_3^2 + \beta_3^2 + \gamma_3^2 \end{vmatrix}.$$

§ 70. If one of the two determinants, whose product in determinant form is wished, be of a lower order than the other, we can raise its order to that of the other in the manner already shown (§ 42), and then proceed as before. As, however, this preliminary change may be accomplished in a variety of ways, there thus arises an increased variety in the possible forms of the result. For example, if the product wanted be that of $|a_1b_2c_3d_4|$ and $|\alpha_1\beta_2|$, we have

$$|a_1b_2c_3d_4|\,|\alpha_1\beta_2|$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ \alpha_2 & \beta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & c_1 & d_1 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & c_2 & d_2 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & c_3 & d_3 \\ a_4\alpha_1 + b_4\beta_1 & a_4\alpha_2 + b_4\beta_2 & c_4 & d_4 \end{vmatrix}; \quad \text{or}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & z_1 & 0 \\ \alpha_2 & \beta_2 & z_2 & 0 \\ 0 & 0 & 1 & 0 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix},$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1z_1, & a_1\alpha_2 + b_1\beta_2 + c_1z_2, & c_1, & a_1x_4 + b_1y_4 + c_1z_4 + d_1 \\ a_2\alpha_1 + b_2\beta_1 + c_2z_1, & a_2\alpha_2 + b_2\beta_2 + c_2z_2, & c_2, & a_2x_4 + b_2y_4 + c_2z_4 + d_2 \\ a_3\alpha_1 + b_3\beta_1 + c_3z_1, & a_3\alpha_2 + b_3\beta_2 + c_3z_2, & c_3, & a_3x_4 + b_3y_4 + c_3z_4 + d_3 \\ a_4\alpha_1 + b_4\beta_1 + c_4z_1, & a_4\alpha_2 + b_4\beta_2 + c_4z_2, & c_4, & a_4x_4 + b_4y_4 + c_4z_4 + d_4 \end{vmatrix}.$$

§ 71. If the process of § 67 be followed to find the product of two determinants, one or both of which contain one or more zero columns, there results a determinant whose value might not otherwise readily appear, but which, from viewing it as arising in the manner stated, we know must equal zero. Thus the determinant

$$\begin{vmatrix} a_1x_1 + b_1y_1 & a_1x_2 + b_1y_2 & a_1x_3 + b_1y_3 \\ a_2x_1 + b_2y_1 & a_2x_2 + b_2y_2 & a_2x_3 + b_2y_3 \\ a_3x_1 + b_3y_1 & a_3x_2 + b_3y_2 & a_3x_3 + b_3y_3 \end{vmatrix} = 0,$$

since (§ 67) it is equal to

$$\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & \lambda_1 \\ x_2 & y_2 & \lambda_2 \\ a_3 & y_3 & \lambda_3 \end{vmatrix}.$$

§ 72. *If there be two sets of elements both consisting of* n *rows of* r *elements* (r *being greater than* n), *and a determinant be formed from them in the way in which the product of two determinants is formed, then this determinant is equal to the sum of every product whose first factor is a determinant obtained by taking* n *columns of* *the first set of elements, and whose other factor is the determinant obtained by taking the corresponding* n *columns of the second set.*

Let the two sets of elements be

$$\begin{array}{llll} a_{11} & a_{12} \ldots\ldots a_{1n} & \ldots\ldots & a_{1r} \\ a_{21} & a_{22} \ldots\ldots a_{2n} & \ldots\ldots & a_{2r} \\ \multicolumn{4}{c}{\dotfill} \\ a_{n1} & a_{n2} \ldots\ldots a_{nn} & \ldots\ldots & a_{nr} \end{array} \quad \text{and} \quad \begin{array}{llll} b_{11} & b_{12} \ldots\ldots b_{1n} & \ldots\ldots & b_{1r} \\ b_{21} & b_{22} \ldots\ldots b_{2n} & \ldots\ldots & b_{2r} \\ \multicolumn{4}{c}{\dotfill} \\ b_{n1} & b_{n2} \ldots\ldots b_{nn} & \ldots\ldots & b_{nr}, \end{array}$$

so that the determinant referred to, Δ say, is

$$\begin{vmatrix} a_{11}b_{11}+\ldots+a_{1r}b_{1r} & a_{11}b_{21}+\ldots+a_{1r}b_{2r} & \ldots\ldots & a_{11}b_{n1}+\ldots+a_{1r}b_{nr} \\ a_{21}b_{11}+\ldots+a_{2r}b_{1r} & a_{21}b_{21}+\ldots+a_{2r}b_{2r} & \ldots\ldots & a_{21}b_{n1}+\ldots+a_{2r}b_{nr} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1}b_{11}+\ldots+a_{nr}b_{1r} & a_{n1}b_{21}+\ldots+a_{nr}b_{2r} & \ldots\ldots & a_{n1}b_{n1}+\ldots+a_{nr}b_{nr} \end{vmatrix}.$$

Fixing upon any n terms of the first element of Δ, let us delete the other $r-n$ terms and the corresponding $r-n$ terms in all the other elements, and call the resulting determinant ∂_1: again fixing upon any other n terms of the first element of Δ, let us proceed in the same way, and call the resulting determinant ∂_2: and so on. For example, if it be the *last* $r-n$ terms which are deleted in obtaining ∂_1,

$$\partial_1 = \begin{vmatrix} a_{11}b_{11}+\ldots+a_{1n}b_{1n} & a_{11}b_{21}+\ldots+a_{1n}b_{2n} & \ldots\ldots & a_{11}b_{n1}+\ldots+a_{1n}b_{nn} \\ a_{21}b_{11}+\ldots+a_{2n}b_{1n} & a_{21}b_{21}+\ldots+a_{2n}b_{2n} & \ldots\ldots & a_{21}b_{n1}+\ldots+a_{2n}b_{nn} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1}b_{11}+\ldots+a_{nn}b_{1n} & a_{n1}b_{21}+\ldots+a_{nn}b_{2n} & \ldots\ldots & a_{n1}b_{n1}+\ldots+a_{nn}b_{nn} \end{vmatrix}.$$

The number of such determinants is evidently the number of combinations of r things taken n at a time, *i.e.*,

$$\frac{r(r-1)\ldots(r-n+1)}{n!}.$$

Returning to Δ we see that as each element consists of r terms and there are n columns, Δ may be partitioned into r^n determinants, the columns of each of which are got by taking from each of the columns of Δ a set of terms in the same vertical line. Of these determinants, however, those which do not vanish are in number only the number of permutations of r things taken n at a time, so that Δ is equal to the sum of $r(r-1)\dots(r-n+1)$ determinants with monomial elements.

Now each of these monomial-element determinants is part of one of the determinants of the ∂ series, viz. that one in which its own columns occur as parts of columns. For example, $|(a_{11}b_{11})\,(a_{22}b_{22})\dots(a_{nn}b_{nn})|$ is a part of ∂_1, being one of the monomial-element determinants into which ∂_1 can be partitioned. In other words, the monomial-element determinant which is formed by taking from the first column of Δ the set of x^{th} terms, from the second column of Δ the set of y^{th} terms, and so on, is part of that member of the ∂ series which is obtained by retaining the x^{th}, y^{th}, ... terms of each element of Δ and deleting the rest. Hence Δ is a part of $\partial_1+\partial_2+\dots$. But when each member of the ∂ series is partitioned like Δ into monomial-element determinants, the number of these which do not vanish is (p. 96) $n!$ for each case, and therefore for the whole series is

$$n! \times \frac{r(r-1)\dots\dots(r-n+1)}{n!} \quad \text{or} \quad r(r-1)\dots\dots(r-n+1),$$

—that is to say, exactly the number of non-evanescent monomial-element determinants into which Δ can be partitioned.

$$\therefore\ \Delta = \partial_1+\partial_2+\partial_3+\dots\dots$$

Now (§ 67) $\partial_1, \partial_2, \dots\dots$ are each the product of two determinants, viz. that member of the ∂ series which is obtained by retaining the x^{th}, y^{th}, ... terms of each element of Δ and deleting the rest is the product of the two determinants whose columns are the x^{th}, y^{th}, ... columns of the first and second given set of elements respectively. Thus the theorem is established.

EXAMPLE:—Formed from the two sets of elements

$$\begin{matrix} a & b & c \\ a' & b' & c', \end{matrix} \qquad \begin{matrix} x & y & z \\ x' & y' & z', \end{matrix}$$

the determinant

$$\begin{vmatrix} ax+by+cz & ax'+by'+cz' \\ a'x+b'y+c'z & a'x'+b'y'+c'z' \end{vmatrix}$$

$$= \begin{vmatrix} ax+by & ax'+by' \\ a'x+b'y & a'x'+b'y' \end{vmatrix} + \begin{vmatrix} ax+cz & ax'+cz' \\ a'x+c'z & a'x'+c'z' \end{vmatrix} + \begin{vmatrix} by+cz & by'+cz' \\ b'y+c'z & b'y'+c'z' \end{vmatrix},$$

$$= \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \times \begin{vmatrix} x & y \\ x' & y' \end{vmatrix} + \begin{vmatrix} a & c \\ a' & c' \end{vmatrix} \times \begin{vmatrix} x & z \\ x' & z' \end{vmatrix} + \begin{vmatrix} b & c \\ b' & c' \end{vmatrix} \times \begin{vmatrix} y & z \\ y' & z' \end{vmatrix}.$$

Although here the determinant with trinomial elements is expressed as the sum of three determinants with binomial elements, it must be noticed that this is not possible in the case of *every* determinant with trinomial elements. The general theorem in fact is

$$\begin{vmatrix} a_1+b_1+c_1 & x_1+y_1+z_1 \\ a_2+b_2+c_2 & x_2+y_2+z_2 \end{vmatrix} = \left.\begin{aligned} &\begin{vmatrix} a_1+b_1 & x_1+y_1 \\ a_2+b_2 & x_2+y_2 \end{vmatrix} + \begin{vmatrix} a_1+c_1 & x_1+z_1 \\ a_2+c_2 & x_2+z_2 \end{vmatrix} + \begin{vmatrix} b_1+c_1 & y_1+z_1 \\ b_2+c_2 & y_2+z_2 \end{vmatrix} \\ &\qquad - \begin{vmatrix} a_1 & x_1 \\ a_2 & x_2 \end{vmatrix} - \begin{vmatrix} b_1 & y_1 \\ b_2 & y_2 \end{vmatrix} - \begin{vmatrix} c_1 & z_1 \\ c_2 & z_2 \end{vmatrix} \end{aligned}\right\}.$$

It is also worthy of note that the *minors* of the determinant product of § 67 have the form of the determinant now dealt with in § 72.

§ 73. Having found as in § 67 the product of two determinants, the product of the result and another determinant may be similarly found, and thus we see generally that *the product of any number of determinants of the same or different orders is obtainable as a determinant of the order which is highest among the factors.*

EXAMPLE 1. Show that

$$\begin{vmatrix} y^3 & -y^2x & yx^2 & -x^3 \\ a & b & c & d \\ b & c & d & e \\ c & d & e & f \end{vmatrix} = \begin{vmatrix} ax+by & bx+cy & cx+dy \\ bx+cy & cx+dy & dx+ey \\ cx+dy & dx+ey & ex+fy \end{vmatrix}.$$

By § 67 we have

$$\begin{vmatrix} y^3 & -y^2x & yx^2 & -x^3 \\ a & b & c & d \\ b & c & d & e \\ c & d & e & f \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & y \end{vmatrix} = \begin{vmatrix} y^3 & 0 & 0 & 0 \\ a & ax+by & bx+cy & cx+dy \\ b & bx+cy & cx+dy & dx+ey \\ c & cx+dy & dx+ey & ex+fy \end{vmatrix},$$

$$= y^3 \begin{vmatrix} ax+by & bx+cy & cx+dy \\ bx+cy & cx+dy & dx+ey \\ cx+dy & dx+ey & ex+fy \end{vmatrix},$$

whence by division the identity is established.

EXAMPLE 2. Prove that if the expression

$$ax^2+by^2+cz^2+dxy+eyz+fzx+gx+hy+kz+l$$

be the product of two linear factors, $\alpha_1x+\beta_1y+\gamma_1z+\delta_1$ and $\alpha_2x+\beta_2y+\gamma_2z+\delta_2$ say, then

$$\begin{vmatrix} 2a & d & f & g \\ d & 2b & e & h \\ f & e & 2c & k \\ g & h & k & 2l \end{vmatrix} = 0.$$

Multiplying the factors together and comparing the result with the given expression, we have

$$\begin{aligned} a &= \alpha_1\alpha_2, & f &= \alpha_1\gamma_2+\alpha_2\gamma_1, \\ b &= \beta_1\beta_2, & g &= \alpha_1\delta_2+\alpha_2\delta_1, \\ c &= \gamma_1\gamma_2, & h &= \beta_1\delta_2+\beta_2\delta_1, \\ d &= \alpha_1\beta_2+\alpha_2\beta_1, & k &= \gamma_1\delta_2+\gamma_2\delta_1, \\ e &= \beta_1\gamma_2+\beta_2\gamma_1, & l &= \delta_1\delta_2. \end{aligned}$$

Hence

$$\begin{vmatrix} 2a & d & f & g \\ d & 2b & e & h \\ f & e & 2c & k \\ g & h & k & 2l \end{vmatrix} = \begin{vmatrix} \alpha_1\alpha_2+\alpha_2\alpha_1 & \alpha_1\beta_2+\alpha_2\beta_1 & \alpha_1\gamma_2+\alpha_2\gamma_1 & \alpha_1\delta_2+\alpha_2\delta_1 \\ \alpha_1\beta_2+\alpha_2\beta_1 & \beta_1\beta_2+\beta_2\beta_1 & \beta_1\gamma_2+\beta_2\gamma_1 & \beta_1\delta_2+\beta_2\delta_1 \\ \alpha_1\gamma_2+\alpha_2\gamma_1 & \beta_1\gamma_2+\beta_2\gamma_1 & \gamma_1\gamma_2+\gamma_2\gamma_1 & \gamma_1\delta_2+\gamma_2\delta_1 \\ \alpha_1\delta_2+\alpha_2\delta_1 & \beta_1\delta_2+\beta_2\delta_1 & \gamma_1\delta_2+\gamma_2\delta_1 & \delta_1\delta_2+\delta_2\delta_1 \end{vmatrix},$$

$$\text{i.e.}\quad \begin{vmatrix} 2a & d & f & g \\ d & 2b & e & h \\ f & e & 2c & k \\ g & h & k & 2l \end{vmatrix} = \begin{vmatrix} \alpha_1 & \alpha_2 & 0 & 0 \\ \beta_1 & \beta_2 & 0 & 0 \\ \gamma_1 & \gamma_2 & 0 & 0 \\ \delta_1 & \delta_2 & 0 & 0 \end{vmatrix} \times \begin{vmatrix} \alpha_2 & \alpha_1 & 0 & 0 \\ \beta_2 & \beta_1 & 0 & 0 \\ \gamma_2 & \gamma_1 & 0 & 0 \\ \delta_2 & \delta_1 & 0 & 0 \end{vmatrix}, \qquad (\S\,67)$$

$$= 0.$$

Not only does this determinant (known as the *discriminant* of $ax^2+by^2+cz^2+dxy+eyz+fzx+gx+hy+kz+l$) itself vanish in the circumstances mentioned, but so also do all its principal minors. Thus, taking the minor obtained by deleting the *second* row and *third* column, we find

$$\begin{vmatrix} 2a & d & g \\ f & e & k \\ g & h & 2l \end{vmatrix} = \begin{vmatrix} \alpha_1 & \alpha_2 & 0 \\ \gamma_1 & \gamma_2 & 0 \\ \delta_1 & \delta_2 & 0 \end{vmatrix} \begin{vmatrix} \alpha_2 & \alpha_1 & 0 \\ \beta_2 & \beta_1 & 0 \\ \delta_2 & \delta_1 & 0 \end{vmatrix} = 0;$$

the first factor here being obtainable from the first factor above by deleting the *second* row and fourth column, and the second factor from the second factor above by deleting the *third* row and fourth column.

EXAMPLE 3. Show that if in

$$ax^2+by^2+cz^2+dxy+eyz+fzx+gx+hy+kz+l$$

we put

$$\begin{aligned} x &= \alpha_1 X+\beta_1 Y+\gamma_1 Z, \\ y &= \alpha_2 X+\beta_2 Y+\gamma_2 Z, \\ z &= \alpha_3 X+\beta_3 Y+\gamma_3 Z, \end{aligned}$$

arrange the result with regard to X, Y, Z as the given expression is arranged with regard to x, y, z, and denote the coefficients in order by A, B, C, ..., then

$$\begin{vmatrix} 2A & D & F & G \\ D & 2B & E & H \\ F & E & 2C & K \\ G & H & K & 2L \end{vmatrix} = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}^2 \begin{vmatrix} 2a & d & f & g \\ d & 2b & e & h \\ f & e & 2c & k \\ g & h & k & 2l \end{vmatrix}.$$

Making the substitutions and arranging, we find

$$\begin{aligned} A &= a\alpha_1^2+b\alpha_2^2+c\alpha_3^2+d\alpha_1\alpha_2+e\alpha_2\alpha_3+f\alpha_1\alpha_3, & G &= g\alpha_1+h\alpha_2+k\alpha_3, \\ B &= a\beta_1^2+b\beta_2^2+c\beta_3^2+d\beta_1\beta_2+e\beta_2\beta_3+f\beta_1\beta_3, & H &= g\beta_1+h\beta_2+k\beta_3, \\ C &= a\gamma_1^2+b\gamma_2^2+c\gamma_3^2+d\gamma_1\gamma_2+e\gamma_2\gamma_3+f\gamma_1\gamma_3, & K &= g\gamma_1+h\gamma_2+h\gamma_3, \end{aligned}$$

$$\begin{aligned} D &= 2a\alpha_1\beta_1+2b\alpha_2\beta_2+2c\alpha_3\beta_3+d\alpha_2\beta_1+d\alpha_1\beta_2+e\alpha_3\beta_2+e\alpha_2\beta_3+f\alpha_3\beta_1+f\alpha_1\beta_3, \\ E &= 2a\beta_1\gamma_1+2b\beta_2\gamma_2+2c\beta_3\gamma_3+d\beta_2\gamma_1+d\beta_1\gamma_2+e\beta_3\gamma_2+e\beta_2\gamma_3+f\beta_3\gamma_1+f\beta_1\gamma_3, \\ F &= 2a\gamma_1\alpha_1+2b\gamma_2\alpha_2+2c\gamma_3\alpha_3+d\gamma_2\alpha_1+d\gamma_1\alpha_2+e\gamma_3\alpha_2+e\gamma_2\alpha_3+f\gamma_3\alpha_1+f\gamma_1\alpha_3. \end{aligned}$$

Now

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}^2 \begin{vmatrix} 2a & d & f & g \\ d & 2b & e & h \\ f & e & 2c & k \\ g & h & k & 2l \end{vmatrix} = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ \beta_1 & \beta_2 & \beta_3 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}^2 \begin{vmatrix} 2a & d & f & g \\ d & 2b & e & h \\ f & e & 2c & k \\ g & h & k & 2l \end{vmatrix},$$

$$= |\alpha_1\beta_2\gamma_3| \begin{vmatrix} 2a\alpha_1+d\alpha_2+f\alpha_3 & d\alpha_1+2b\alpha_2+e\alpha_3 & f\alpha_1+e\alpha_2+2c\alpha_3 & g\alpha_1+h\alpha_2+k\alpha_3 \\ 2a\beta_1+d\beta_2+f\beta_3 & d\beta_1+2b\beta_2+e\beta_3 & f\beta_1+e\beta_2+2c\beta_3 & g\beta_1+h\beta_2+k\beta_3 \\ 2a\gamma_1+d\gamma_2+f\gamma_3 & d\gamma_1+2b\gamma_2+e\gamma_3 & f\gamma_1+e\gamma_2+2c\gamma_3 & g\gamma_1+h\gamma_2+k\gamma_3 \\ g & h & k & 2l \end{vmatrix},$$

$$= \begin{vmatrix} 2A & D & F & G \\ D & 2B & E & H \\ F & E & 2C & K \\ G & H & K & 2L \end{vmatrix},$$

if in multiplying we use $|\alpha_1\beta_2\gamma_3|$ again in its altered form.

It is also apparent from this that the complementary minors of $2L$ and $2l$ are connected in the same way by the multiplier $|\alpha_1\beta_2\gamma_3|^2$.

Exercises. Set XII.

Perform the following multiplications, giving the results as determinants:—

1. $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} |a_1b_2c_3|.$ 2. $\begin{vmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & f \end{vmatrix} \begin{vmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{vmatrix}.$

3. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \begin{vmatrix} a^2 & -a & 1 \\ b^2 & -b & 1 \\ c^2 & -c & 1 \end{vmatrix}.$ 4. $\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} \begin{vmatrix} a+b+\frac{1}{2}c & -\frac{1}{2}a & -\frac{1}{2}b \\ -\frac{1}{2}c & b+c+\frac{1}{2}a & -\frac{1}{2}b \\ -\frac{1}{2}c & -\frac{1}{2}a & c+a+\frac{1}{2}b \end{vmatrix}.$

5. By changing x^2+y^2, y^2+z^2, z^2+x^2 into determinant form and multiplying, find an expression for their product as the sum of two squares.

6. Find the product of

$$\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix},$$

and thence resolve the former determinant into simple factors.

7. Prove the identity of Ex. 14, Set VIII., by using

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$$

as a multiplier. Write down the corresponding multiplier in the case of Ex. 15, Set VIII.

Give the quotients in the following cases as determinants :—

8. $$\begin{vmatrix} 3\alpha^2 & \beta^2+\beta\alpha+\alpha^2 & \gamma^2+\gamma\alpha+\alpha^2 \\ \beta^2+\beta\alpha+\alpha^2 & 3\beta^2 & \gamma^2+\gamma\beta+\beta^2 \\ \gamma^2+\gamma\alpha+\alpha^2 & \gamma^2+\gamma\beta+\beta^2 & 3\gamma^2 \end{vmatrix} \div \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}.$$

9. $$\begin{vmatrix} 2xy & y^2+x & x^2+y \\ x^2+y & 2xy & x+y^2 \\ x+y^2 & y+x^2 & 2xy \end{vmatrix} \div \begin{vmatrix} x & y & 0 \\ y & 0 & x \\ 0 & x & y \end{vmatrix}.$$

10. Use the multiplication theorem to find the simple factors of the determinant of Ex. 24, Set VII.

11. Find the product of

$$\begin{vmatrix} a+b\sqrt{-1} & -c+d\sqrt{-1} \\ c+d\sqrt{-1} & a-b\sqrt{-1} \end{vmatrix} \text{ and } \begin{vmatrix} \alpha+\beta\sqrt{-1} & -\gamma+\delta\sqrt{-1} \\ \gamma+\delta\sqrt{-1} & \alpha-\beta\sqrt{-1} \end{vmatrix};$$

and thence show how the product of two sums of four squares is itself expressible as a sum of four squares.

12. Show that

$$2(a+b+c)\begin{vmatrix} a & b & c \\ b & c & a \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a^2 & b^2 & c^2 \\ b & c & a \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} a & b & c \\ b^2 & c^2 & a^2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} ab & bc & ca \\ b & c & a \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} a & b & c \\ ab & bc & ca \\ 1 & 1 & 1 \end{vmatrix}.$$

13. Show that the identity of Ex. 12, Set VII., follows from finding the product of

$$\begin{vmatrix} a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ a_3 & b_3 & c_3 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 & k_1 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & k_3 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ h_1 & h_2 & h_3 & 1 \end{vmatrix}.$$

14. Find the expansion of

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} \times \begin{vmatrix} a+\lambda & h & g \\ h & b+\lambda & f \\ g & f & c+\lambda \end{vmatrix}$$

according to descending powers of λ, showing that the coefficients are alternately negative and positive.

15. Find the product of

$$\begin{vmatrix} a_1 & a_2 & a_3 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} b_2 & b_3 & 0 & 0 \\ b_3 & 0 & 0 & 0 \\ -a_3 & 0 & -1 & 0 \\ -a_2 & -a_3 & 0 & -1 \end{vmatrix},$$

and thence show that the former determinant is equal to

$$\begin{Vmatrix} |a_1\ b_2| & |a_1\ b_3| \\ |a_1\ b_3| & |a_2\ b_3| \end{Vmatrix}.$$

Resolve into determinant factors—

16. $$\begin{vmatrix} a^2+bc & ab & bd \\ ac & bc+de & df \\ ce & ef & de+f^2 \end{vmatrix}.$$

17. $$\begin{vmatrix} a_1 & b_1x_1+c_1y_1 & b_1x_2+c_1y_2 \\ a_2 & b_2x_1+c_2y_1 & b_2x_2+c_2y_2 \\ a_3 & b_3x_1+c_3y_1 & b_3x_2+c_3y_2 \end{vmatrix}.$$

18. $$\begin{vmatrix} ax_1+cz_1 & 0 & fx_1+gz_1 \\ ax_2+by_2+cz_2 & dy_2 & fx_2+gz_2 \\ by_3+cz_3 & dy_3 & gz_3 \end{vmatrix}.$$

19. Prove that

$$-4\begin{vmatrix} 1 & x_1 & y_1 & x_1^2+y_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2 \\ 1 & x_3 & y_3 & x_3^2+y_3^2 \\ 1 & x_4 & y_4 & x_4^2+y_4^2 \end{vmatrix}^2$$

$$=\begin{vmatrix} 0 & (x_1-x_2)^2+(y_1-y_2)^2 & (x_1-x_3)^2+(y_1-y_3)^2 & (x_1-x_4)^2+(y_1-y_4)^2 \\ (x_1-x_2)^2+(y_1-y_2)^2 & 0 & (x_2-x_3)^2+(y_2-y_3)^2 & (x_2-x_4)^2+(y_2-y_4)^2 \\ (x_1-x_3)^2+(y_1-y_3)^2 & (x_2-x_3)^2+(y_2-y_3)^2 & 0 & (x_3-x_4)^2+(y_3-y_4)^2 \\ (x_1-x_4)^2+(y_1-y_4)^2 & (x_2-x_4)^2+(y_2-y_4)^2 & (x_3-x_4)^2+(y_3-y_4)^2 & 0 \end{vmatrix}.$$

20. Prove, as in Ex. 15, that

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & b_1 & b_2 & b_3 & b_4 \\ 0 & b_1 & b_2 & b_3 & b_4 & 0 \\ b_1 & b_2 & b_3 & b_4 & 0 & 0 \end{vmatrix} = \begin{vmatrix} |a_1\ b_2| & |a_1\ b_3| & |a_1\ b_4| \\ |a_1\ b_3| & |a_1\ b_4|+|a_2\ b_3| & |a_2\ b_4| \\ |a_1\ b_4| & |a_2\ b_4| & |a_3\ b_4| \end{vmatrix}.$$

21. Prove, in the same way, that

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & b_1 & b_2 & b_3 \\ 0 & b_1 & b_2 & b_3 & 0 \\ b_1 & b_2 & b_3 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} |a_1 b_2| & |a_1 b_3| & -a_4 b_1 \\ |a_1 b_3| & -a_4 b_1 + |a_2 b_3| & -a_4 b_2 \\ -a_4 b_1 & -a_4 b_2 & -a_4 b_3 \end{vmatrix} \div (-a_4) = \begin{vmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_1 b_2 & |a_1 b_3| + a_2 b_2 & a_2 b_3 - a_4 b_1 \\ a_1 b_3 & a_2 b_3 - a_4 b_1 & a_3 b_3 - a_4 b_2 \end{vmatrix} \div a_1.$$

22. Show that the proposition of § 53 may be established by using the multiplication theorem.

23. Find the quotient of

$$\begin{vmatrix} (s-a_1)^2 & a_1^2 & a_1^2 & \ldots\ldots & a_1^2 \\ a_2^2 & (s-a_2)^2 & a_2^2 & \ldots\ldots & a_2^2 \\ a_3^2 & a_3^2 & (s-a_3)^2 & \ldots\ldots & a_3^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n^2 & a_n^2 & a_n^2 & \ldots\ldots & (s-a_n)^2 \end{vmatrix} \div \begin{vmatrix} s-a_1 & a_1 & a_1 & \ldots\ldots & a_1 \\ a_2 & s-a_2 & a_2 & \ldots\ldots & a_2 \\ a_3 & a_3 & s-a_3 & \ldots\ldots & a_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & a_n & a_n & \ldots\ldots & s-a_n \end{vmatrix},$$

where $s = a_1 + a_2 + \ldots. + a_n$.

24. Prove that

$$\begin{vmatrix} 0 & \alpha_2(\alpha_1\beta + \alpha\beta_1) - 2\beta_2\alpha\alpha_1 & \alpha_2(\alpha_1\gamma + \alpha\gamma_1) - 2\gamma_2\alpha\alpha_1 \\ \beta_2(\beta\alpha_1 + \beta_1\alpha) - 2\alpha_2\beta\beta_1 & 0 & \beta_2(\beta_1\gamma + \beta\gamma_1) - 2\gamma_2\beta\beta_1 \\ \gamma_2(\gamma_1\alpha + \gamma\alpha_1) - 2\alpha_2\gamma\gamma_1 & \gamma_2(\gamma_1\beta + \gamma\beta_1) - 2\beta_2\gamma\gamma_1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \alpha\beta_1\gamma_1 + \alpha_1\beta_1\gamma - 2\alpha_1\beta\gamma_1 & \alpha\beta_2\gamma_2 + \alpha_2\beta_2\gamma - 2\alpha_2\beta\gamma_2 \\ \alpha_1\beta\gamma + \alpha\beta\gamma_1 - 2\alpha\beta_1\gamma & 0 & \alpha_1\beta_2\gamma_2 + \alpha_2\beta_2\gamma_1 - 2\alpha_2\beta_1\gamma_2 \\ \alpha_2\beta\gamma + \alpha\beta\gamma_2 - 2\alpha\beta_2\gamma & \alpha_2\beta_1\gamma_1 + \alpha_1\beta_1\gamma_2 - 2\alpha_1\beta_2\gamma_1 & 0 \end{vmatrix}.$$

Establish the three following identities:—

25.
$$\begin{vmatrix} a+b+c+\tfrac{1}{2}d & -\tfrac{1}{2}a & -\tfrac{1}{2}b & -\tfrac{1}{2}c \\ -\tfrac{1}{2}d & b+c+d+\tfrac{1}{2}a & -\tfrac{1}{2}b & -\tfrac{1}{2}c \\ -\tfrac{1}{2}d & -\tfrac{1}{2}a & c+d+a+\tfrac{1}{2}b & -\tfrac{1}{2}c \\ -\tfrac{1}{2}d & -\tfrac{1}{2}a & -\tfrac{1}{2}b & d+a+b+\tfrac{1}{2}c \end{vmatrix} = \tfrac{1}{2}(a+b+c+d)^4.$$

26.
$$\begin{vmatrix} (a+b+c)^2 & d^2 & d^2 & d^2 \\ a^2 & (b+c+d)^2 & a^2 & a^2 \\ b^2 & b^2 & (c+d+a)^2 & b^2 \\ c^2 & c^2 & c^2 & (d+a+b)^2 \end{vmatrix} = 2(a+b+c+d)^4\Sigma a^2bc.$$

27. $$\begin{vmatrix} 3d & s+d & s+d & s+d \\ s+a & 3a & s+a & s+a \\ s+b & s+b & 3b & s+b \\ s+c & s+c & s+c & 3c \end{vmatrix} = -12\Sigma a^2bc,$$

if $s = a+b+c+d$.

§ 74. *If, in a determinant of the* nth *order the minor formed of the elements common to the first* m *rows and the first* m *columns be multiplied by its complementary, the product gives* m! (n – m)! *terms of the original determinant.*

Let the determinant be

$$\begin{vmatrix} a_1 & a_2 & a_3 \ldots\ldots a_m & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_1 & b_2 & b_3 \ldots\ldots b_m & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ l_1 & l_2 & l_3 \ldots\ldots l_m & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots & p_{m+1} \quad p_{m+2} \ldots\ldots p_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots & q_{m+1} \quad q_{m+2} \ldots\ldots q_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots & x_{m+1} \quad x_{m+2} \ldots\ldots x_n \end{vmatrix} \text{ or } D, \text{ say},$$

where no elements are represented unless a few of those in the two minors referred to, and denote the minor

$$|a_1 b_2 \ldots\ldots l_m| \text{ by } \mathfrak{m}$$

and its complementary

$$|p_{m+1} q_{m+2} \ldots\ldots x_n| \text{ by } \mathfrak{n}.$$

If any term of $\mathfrak{m}$ be taken and any term of $\mathfrak{n}$, the product of the two terms must contain one and only one element from each row of D, and one and only one element from each column, and must therefore, setting aside the question of sign, be a term of D.

Further, suppose that the numbers whose inversions of order fix the sign of the term taken from $\mathfrak{m}$ are

$$\beta,\ \delta,\ \gamma,\ \ldots,\ \kappa,$$

the number of the said inversions being r; and suppose that the corresponding numbers in the case of the term taken from $\mathfrak{n}$ are

$$\xi,\ \pi,\ \rho,\ \ldots,\ \tau,$$

the number of inversions being s. Then the sign-factor of the product of the two terms would be

$$(-1)^r \times (-1)^s.$$

But the series of numbers for fixing the sign of it viewed as a term of D would be

$$\beta,\ \delta,\ \gamma,\ \ldots,\ \kappa,\ m+\xi,\ m+\pi,\ m+\rho,\ \ldots,\ m+\tau;$$

and as each of the numbers $m+\xi$, $m+\pi$, …, $m+\tau$ is greater than any one of the numbers β, δ, …, κ, and the number of inversions in $m+\xi$, $m+\pi$, …, $m+\tau$ is the same as in ξ, π, …, τ, the total number of inversions in the series must therefore be $r+s$. Consequently the sign-factor of the product viewed as a term of D would be

$$(-1)^{r+s},$$

which, as we have just seen, is the sign it actually bears.

Thus the product of any term of $\mathfrak{m}$ and any term of $\mathfrak{n}$ is a term of D; therefore if $\mathfrak{m}$ which consists of $m!$ terms be multiplied by $\mathfrak{n}$ which consists of $(n-m)!$ terms, there will result $m!\,(n-m)!$ terms of D.

§ 75. *If, in a determinant of the* nth *order, the minor formed of the elements common to* m *rows, viz. the* hth, kth, lth, …., *and* m *columns, viz. the* rth, sth, uth, …., *be multi-*

plied by its complementary, the product taken with the sign $(-1)^{h+k+l+\dots+r+s+u+\dots}$ *gives* m! (n − m)! *terms of the original determinant.*

Let the determinant be D, the minor $\mathfrak{m}$, and its complementary $\mathfrak{n}$.

Making the h^{th} row pass over the $h-1$ rows which precede it, then the k^{th} row over $k-2$ preceding rows, then the l^{th} row over $l-3$ preceding rows, and so on, we have a determinant D' whose first m rows are the h^{th}, k^{th}, l^{th}, ... of D, and such that

$$D' = (-1)^{h-1+k-2+l-3+\dots} \times D.$$

Again, by treating the r^{th}, s^{th}, u^{th}, ... columns of this determinant in like fashion, we obtain a third determinant D'', whose first m rows and first m columns are the m rows and m columns of D out of which $\mathfrak{m}$ is formed, and such that

$$\begin{aligned} D'' &= (-1)^{h-1+k-2+l-3+\dots} \times (-1)^{r-1+s-2+u-3+\dots} \times D, \\ \text{or} \quad &= (-1)^{h+k+l+\dots+r+s+u+\dots} \times D, \end{aligned}$$

since $-1-2-3-\dots-1-2-3-\dots$ is even. Now (§ 74) the product $\mathfrak{m} \times \mathfrak{n}$ gives $m!(n-m)!$ terms of D''; therefore, taken with the sign-factor $(-1)^{h+k+l+\dots+r+s+u+\dots}$ it will give $m!(n-m)!$ terms of D.

Example. Taking the first, second, fifth rows, and the third, fourth, fifth columns of

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix},$$

we have the minor

$$\begin{vmatrix} a_3 & a_4 & a_5 \\ b_3 & b_4 & b_5 \\ e_3 & e_4 & e_5 \end{vmatrix},$$

whose complementary is

$$\begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix}$$

hence $(-1)^{1+2+5+3+4+5} |a_3b_4e_5| \, |c_1d_2|$ gives 3! 2! terms of $|a_1b_2c_3d_4e_5|$. Similarly, $|c_1d_2e_3| \, |a_4b_5|$ and $-|b_2c_3d_5| \, |a_1e_4|$ each give twelve terms: and, as we already know (§ 46), $-b_3|a_1c_2d_4e_5|$ and $e_1|a_2b_3c_4d_5|$ each give twenty-four.

§ 76. The proposition of § 75 is equivalent to the statement that the co-factor of a minor of a determinant is the complementary minor, the sign to be taken with the product being + or − according as the sum of the numbers indicating the deleted rows and columns is even or odd. As however the sign of the product of the two minors must be the same as the sign of the product of their principal terms, it is evident that we have another and perhaps easier mode of fixing it, viz. by determining the sign of the latter product as a term of the original determinant. Thus, taking the first example above, the sign to be prefixed to $|a_3b_4e_5| \, |c_1d_2|$ is the sign of $a_3b_4e_5c_1d_2$ or $a_3b_4c_1d_2e_5$ as a term of $|a_1b_2c_3d_4e_5|$, and therefore (§ 14) is $(-1)^4$.

§ 77. *If any* m *rows of a determinant be selected and every possible minor of the* m*th* *order be formed from them, and if each be multiplied by its complementary and the sign + or − be affixed to the product according as the sum of the numbers indicating the rows and columns from which the minor is formed be even or odd, the aggregate of the products thus obtained is equal to the original determinant.*

Let the given determinant, D say, be of the nth order, and let the m selected rows be

$$\begin{matrix} a_1 & a_2 & a_3 \, \ldots\ldots \, a_m \, \ldots\ldots \, a_n \\ d_1 & d_2 & d_3 \, \ldots\ldots \, d_m \, \ldots\ldots \, d_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ s_1 & s_2 & s_3 \, \ldots\ldots \, s_m \, \ldots\ldots \, s_n \\ x_1 & x_2 & x_3 \, \ldots\ldots \, x_m \, \ldots\ldots \, x_n . \end{matrix}$$

The number of different minors of the mth order that can be formed from these rows is clearly the number of sets of m columns that can be formed out of n, and therefore is

$$\frac{n!}{m!\,(n-m)!}\,.$$

Now (§ 75) the product of each of these minors and its complementary gives, when its sign is fixed in the manner stated, $m!\,(n-m)!$ terms of D. Using all the different minors, therefore, we obtain

$$m!\,(n-m)! \times \frac{n!}{m!\,(n-m)!} \quad \text{or} \quad n!$$

different terms of D, i.e. the full expansion.

Examples:—Taking the first two rows of

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

we have in all six minors, viz.

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix}, \quad \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \quad \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix}, \quad \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix};$$

hence

$$\begin{aligned} |a_1b_2c_3d_4| = {} & |a_1b_2|\,|c_3d_4| - |a_1b_3|\,|c_2d_4| + |a_1b_4|\,|c_2d_3| + |a_2b_3|\,|c_1d_4| - |a_2b_4|\,|c_1d_3| \\ & + |a_3b_4|\,|c_1d_2|. \end{aligned}$$

By selecting any other pair of rows except the last pair or any pair of columns we should obtain a like development: by selecting one row or three the development is that of § 46.

Similarly, we find

$$\begin{aligned} |a_1b_2c_3d_4e_5| = {} & |a_1b_2c_3|\,|d_4e_5| - |a_1b_2c_4|\,|d_3e_5| + |a_1b_2c_5|\,|d_3e_4| + |a_1b_3c_4|\,|d_2e_5| - |a_1b_3c_5|\,|d_2e_4| \\ & + |a_1b_4c_5|\,|d_2e_3| - |a_2b_3c_4|\,|d_1e_5| + |a_2b_3c_5|\,|d_1e_4| - |a_2b_4c_5|\,|d_1e_3| - |a_3b_4c_5|\,|d_1e_2|. \end{aligned}$$

§ 78. *If any* m *rows of a determinant be selected, and every possible minor of the* m*th* *order be formed from them, and if each minor be multiplied by the complementary of the corresponding minor formed from other* m *rows, and* *the sign + or – be affixed to the product according as the sum of the numbers indicating the rows and columns from which the complementary is formed be even or odd, the aggregate of the products thus obtained is equal to zero.*

Let $|a_{1n}|$ be the determinant, then the aggregate of products referred to is equal (§ 77) to a determinant of the n^{th} order having for m of its rows the m rows from which the first factors are found, and for its other rows the $n-m$ rows from which the second factors are found. But the rows of the latter set cannot be all different from those of the former; for, if from n things a set of m be taken, and from the same n things another set of m, the $n-m$ left the second time must include one or more of those taken the first time. Hence (§ 27) the aggregate of products is equal to zero.

Example. Taking the first and second rows of $|a_1b_2c_3d_4|$, we have the minors

$$|a_1b_2|,\quad |a_1b_3|,\quad |a_1b_4|,\quad |a_2b_3|,\quad |a_2b_4|,\quad |a_3b_4|;$$

and the corresponding minors formed from other two rows, the second and third, being

$$|b_1c_2|,\quad |b_1c_3|,\quad |b_1c_4|,\quad |b_2c_3|,\quad |b_2c_4|,\quad |b_3c_4|,$$

we have as their complementaries

$$|a_3d_4|,\quad |a_2d_4|,\quad |a_2d_3|,\quad |a_1d_4|,\quad |a_1d_3|,\quad |a_1d_2|:$$

then

$$|a_1b_2||a_3d_4| - |a_1b_3||a_2d_4| + |a_1b_4||a_2d_3| + |a_2b_3||a_1d_4| - |a_2b_4||a_1d_3| + |a_3b_4||a_1d_2| = 0,$$

being, in fact, equal to

$$|a_1b_2a_3d_4|.$$

The theorem here exemplified is seen to include that of § 52, as the theorem of § 77 includes that of § 46.

§ 79. *If in any determinant of the* n*th* *order there be* m *rows all having in the same places* n – m *zero-elements, the determinant is expressible as the product of two of its* *minors, viz. the minor, whose elements are the remaining elements of the* m *rows, and its complementary; the sign of the product being + or − according as the sum of the numbers indicating the rows and columns from which the minor is formed be even or odd.*

Seeking to find a development of the determinant as an aggregate of products of complementary minors (§ 77), we see that there are in the m rows only m vertical lines of non-zero elements, and that consequently there can be formed only one non-zero minor of the m^{th} order. The products therefore all vanish except that arising from this minor and its complementary.

EXAMPLE:—

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & 0 & b_3 & 0 & b_5 \\ c_1 & 0 & c_3 & 0 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & 0 & e_3 & 0 & e_5 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ b_1 & 0 & b_3 & 0 & b_5 \\ c_1 & 0 & c_3 & 0 & c_5 \\ e_1 & 0 & e_3 & 0 & e_5 \end{vmatrix} = - \begin{vmatrix} a_2 & a_4 & a_1 & a_3 & a_5 \\ d_2 & d_4 & d_1 & d_3 & d_5 \\ 0 & 0 & b_1 & b_3 & b_5 \\ 0 & 0 & c_1 & c_3 & c_5 \\ 0 & 0 & e_1 & e_3 & e_5 \end{vmatrix},$$

$$= - \begin{vmatrix} a_2 & a_4 \\ d_2 & d_4 \end{vmatrix} \begin{vmatrix} b_1 & b_3 & b_5 \\ c_1 & c_3 & c_5 \\ e_1 & e_3 & e_5 \end{vmatrix}.$$

§ 80. *If, in any determinant of the* n*th* *order, there be* m *rows all having in the same places more than* n − m *zero-elements, the determinant vanishes.*

§ 81. In like manner we see that, conversely, *the product of two determinants of the* r*th* *and* s*th* *orders may be expressed as a determinant of the* (r + s)*th* *order whose elements are* (1) *the* $r^2 + s^2$ *elements of the two determinants, so placed that the said determinants may be complementary minors of the new determinant, and that the sum of the numbers of the rows and columns they occupy may be even,*

(2) rs *zeros completing the rows in which the elements of one of these minors stand, and* (3) *any* rs *finite elements whatever for the remaining places.* Thus

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \times \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & \omega_1 & \omega_2 \\ b_1 & b_2 & \omega_3 & \omega_4 \\ 0 & 0 & x_1 & x_2 \\ 0 & 0 & y_1 & y_2 \end{vmatrix}, = \begin{vmatrix} a_1 & 0 & a_2 & 0 \\ \pi_1 & x_1 & \pi_2 & x_2 \\ b_1 & 0 & b_2 & 0 \\ \pi_3 & y_1 & \pi_4 & y_2 \end{vmatrix}$$

$$= \ldots\ldots\ldots$$

EXAMPLE. Prove that

$$\begin{vmatrix} h_1m_1 & k_1x_1 & h_1n_1 & k_1z_1 \\ h_2m_1 & k_2x_1 & h_2n_1 & k_2z_1 \\ p_1m_2 & q_1x_2 & p_1n_2 & q_1z_2 \\ p_2m_2 & q_2x_2 & p_2n_2 & q_2z_2 \end{vmatrix} = \begin{vmatrix} h_1 & k_1 \\ h_2 & k_2 \end{vmatrix} \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}.$$

From the preceding we have

$$\begin{vmatrix} h_1 & k_1 \\ h_2 & k_2 \end{vmatrix} \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} = \begin{vmatrix} h_1 & k_1 & 0 & 0 \\ h_2 & k_2 & 0 & 0 \\ 0 & 0 & p_1 & q_1 \\ 0 & 0 & p_2 & q_2 \end{vmatrix} \text{ and } \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} = \begin{vmatrix} m_1 & 0 & m_2 & 0 \\ 0 & x_1 & 0 & x_2 \\ n_1 & 0 & n_2 & 0 \\ 0 & z_1 & 0 & z_2 \end{vmatrix},$$

whence by multiplication (§ 67) the desired result is at once obtained.

§ 82. The two different modes which have thus been found for expressing the product of two determinants as a determinant suggest the possibility of deriving the result obtained in the one case (§ 67) from that obtained in the other (§ 81). This can really be done, and the process of transformation is sufficiently instructive to merit attention. Taking the particular case (§ 66) in which the two determinants to be multiplied are of the third order, viz.

$$|A_1B_2C_3| \text{ or } \Delta \text{ and } |a_1b_2c_3| \text{ or } \Delta',$$

we have (§ 81)

$$\Delta\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & A_1 & A_2 & A_3 \\ 0 & 0 & 0 & B_1 & B_2 & B_3 \\ 0 & 0 & 0 & C_1 & C_2 & C_3 \end{vmatrix},$$

where there is specially to be noticed the nine particular elements chosen for the places which may be filled by any nine finite elements whatever. Then, increasing each element of the first column by a_1 times the corresponding element of the fourth column, a_2 times the corresponding element of the fifth column, and a_3 times the corresponding element of the sixth column, and increasing each element of the second column and each element of the third column in a similar fashion, but with the multipliers b_1, b_2, b_3 in the one case, and c_1, c_2, c_3 in the other, the product takes the form

$$\begin{vmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ a_1A_1+a_2A_2+a_3A_3 & b_1A_1+b_2A_2+b_3A_3 & c_1A_1+c_2A_2+c_3A_3 & A_1 & A_2 & A_3 \\ a_1B_1+a_2B_2+a_3B_3 & b_1B_1+b_2B_2+b_3B_3 & c_1B_1+c_2B_2+c_3B_3 & B_1 & B_2 & B_3 \\ a_1C_1+a_2C_2+a_3C_3 & b_1C_1+b_2C_2+b_3C_3 & c_1C_1+c_2C_2+c_3C_3 & C_1 & C_2 & C_3 \end{vmatrix}.$$

Hence (§ 77) it is equal to

$$-\begin{vmatrix} a_1A_1+a_2A_2+a_3A_3 & b_1A_1+b_2A_2+b_3A_3 & c_1A_1+c_2A_2+c_3A_3 \\ a_1B_1+a_2B_2+a_3B_3 & b_1B_1+b_2B_2+b_3B_3 & c_1B_1+c_2B_2+c_3B_3 \\ a_1C_1+a_2C_2+a_3C_3 & b_1C_1+b_2C_2+b_3C_3 & c_1C_1+c_2C_2+c_3C_3 \end{vmatrix}\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix},$$

which becomes at once the result of § 66.

§ 83. In § 67 the product of two determinants of the n^{th} order is given as a determinant of the n^{th} order; in § 81 it is given as a determinant of the $2n^{\text{th}}$ order. We can, however, further express it as a determinant of each of the intermediate orders, so that the forms of §§ 67, 81 may be viewed as the extremes of a series. Thus we have, firstly,

$$|A_1B_2C_3|\,|a_1b_2c_3| = \begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1 & A_2 & A_3 \\ 0 & 0 & 0 & B_1 & B_2 & B_3 \\ 0 & 0 & 0 & C_1 & C_2 & C_3 \end{vmatrix}; \qquad \ldots\ldots(1)$$

secondly, we have

$$|A_1B_2C_3|\,|a_1b_2c_3| = \begin{vmatrix} a_1 & b_1 & c_1 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1 & A_2 & A_3 \\ 0 & 0 & 0 & B_1 & B_2 & B_3 \\ 0 & 0 & 0 & C_1 & C_2 & C_3 \end{vmatrix},$$

$$= \begin{vmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ a_1A_1 & b_1A_1 & c_1A_1 & A_1 & A_2 & A_3 \\ a_1B_1 & b_1B_1 & c_1B_1 & B_1 & B_2 & B_3 \\ a_1C_1 & b_1C_1 & c_1C_1 & C_1 & C_2 & C_3 \end{vmatrix},$$

$$= \begin{vmatrix} a_2 & b_2 & c_2 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 \\ a_1A_1 & b_1A_1 & c_1A_1 & A_2 & A_3 \\ a_1B_1 & b_1B_1 & c_1B_1 & B_2 & B_3 \\ a_1C_1 & b_1C_1 & c_1C_1 & C_2 & C_3 \end{vmatrix}; \qquad \ldots\ldots(2)$$

thirdly, we have

$$|A_1B_2C_3|\,|a_1b_2c_3| = \begin{vmatrix} a_1 & b_1 & c_1 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1 & A_2 & A_3 \\ 0 & 0 & 0 & B_1 & B_2 & B_3 \\ 0 & 0 & 0 & C_1 & C_2 & C_3 \end{vmatrix},$$

$$= \begin{vmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ a_1A_1+a_2A_2 & b_1A_1+b_2A_2 & c_1A_1+c_2A_2 & A_1 & A_2 & A_3 \\ a_1B_1+a_2B_2 & b_1B_1+b_2B_2 & c_1B_1+c_2B_2 & B_1 & B_2 & B_3 \\ a_1C_1+a_2C_2 & b_1C_1+b_2C_2 & c_1C_1+c_2C_2 & C_1 & C_2 & C_3 \end{vmatrix},$$

$$= \begin{vmatrix} a_3 & b_3 & c_3 & 0 \\ a_1A_1+a_2A_2 & b_1A_1+b_2A_2 & c_1A_1+c_2A_2 & A_3 \\ a_1B_1+a_2B_2 & b_1B_1+b_2B_2 & c_1B_1+c_2B_2 & B_3 \\ a_1C_1+a_2C_2 & b_1C_1+b_2C_2 & c_1C_1+c_2C_2 & C_3 \end{vmatrix}; \qquad \ldots\ldots(3)$$

and, fourthly, we have the natural conclusion to these, namely, the procedure and result of § 82.

The general theorem, to which we are in this way led, and which can be proved in the manner indicated, is—

The product of two determinants of the nth *order may be found by substituting zero-columns for* m *columns in the one and for the corresponding* m *columns in the other, multiplying the two determinants thus obtained, increasing the number of the columns in the result by appending in order the deleted columns of the first determinant, increasing the number of rows by superposing the deleted columns of the second determinant after changing them in* *order into rows, putting zeros in the places above the added columns and to the right of the added rows, and prefixing to the determinant thus formed the sign-factor* $(-1)^{m(n+1)}$.

The multiplication-theorem of § 67 is the case of this where $m=0$, although strictly to include it the word "multiplying" here would have to give place to a phrase descriptive of the process intended.

§ 84. Since either of the determinants in each of the products referred to in the theorem of § 77 may itself be expressed as an aggregate of products of complementary minors, we see that by repeated use of the theorem we can develope a determinant as an aggregate of products of *more than two* minors. Thus

$$\begin{aligned}
&|a_1b_2c_3d_4e_5f_6| \\
&\quad = |a_1b_2c_3|\ |d_4e_5f_6| - |a_1b_2c_4|\ |d_3e_5f_6| + \ldots\ldots \\
&\quad = a_1|b_2c_3|\ |d_4e_5f_6| - a_2|b_1c_3|\ |d_4e_5f_6| + a_3|b_1c_2|\ |d_4e_5f_6| \\
&\qquad - a_1|b_2c_4|\ |d_3e_5f_6| + a_2|b_1c_4|\ |d_3e_5f_6| - a_4|b_1c_2|\ |d_3e_5f_6| + \ldots\ldots
\end{aligned}$$

where each term is a product of three minors, one of the first, one of the second, and one of the third order.

§ 85. *The product of two determinants of the same order is equal to the sum of like products obtained from the original by interchanging a particular column of the one determinant with the columns of the other in succession.*

Let $|a_{1n}|$, $|b_{1n}|$ be the two determinants, and, first, let the particular column fixed upon be the first column of $|a_{1n}|$.

If a determinant of the $2n^{\text{th}}$ order be formed, having $|a_{1n}|$ for the minor situated in the first n rows and first n columns, a determinant of n zero elements for the minor situated in the last n rows and first n columns, and $|b_{1n}|$ for the complementary of both of these: and if the first column of $|a_{1n}|$ be then interchanged with the column of zeros below it, we have the determinant

$$\begin{vmatrix} 0 & a_{12} & \dots\dots & a_{1n} & b_{11} & b_{12} & \dots\dots & b_{1n} \\ 0 & a_{22} & \dots\dots & a_{2n} & b_{21} & b_{22} & \dots\dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & a_{n2} & \dots\dots & a_{nn} & b_{n1} & b_{n2} & \dots\dots & b_{nn} \\ a_{11} & 0 & \dots\dots & 0 & b_{11} & b_{12} & \dots\dots & b_{1n} \\ a_{21} & 0 & \dots\dots & 0 & b_{21} & b_{22} & \dots\dots & b_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & 0 & \dots\dots & 0 & b_{n1} & b_{n2} & \dots\dots & b_{nn} \end{vmatrix}, \quad \text{or} \quad \Delta, \text{ say.}$$

Looking to the first n rows of this, and seeking to form from them all the minors of the n^{th} order, with a view to obtain the expansion of the determinant as an aggregate of products of complementary minors, we see that we need only take those minors into which enter the $n-1$ columns surmounting the zeros, for every other minor has a complementary which vanishes. Consequently all the minors worth attending to are got by taking along with these $n-1$ columns the columns of $|b_{1n}|$ in succession: and thus the expansion referred to is

$$(-1)^n \begin{vmatrix} a_{12} \dots a_{1n} & b_{11} \\ a_{22} \dots a_{2n} & b_{21} \\ \dots\dots\dots & \dots \\ a_{n2} \dots a_{nn} & b_{n1} \end{vmatrix} \begin{vmatrix} a_{11} & b_{12} \dots b_{1n} \\ a_{21} & b_{22} \dots b_{2n} \\ \dots & \dots\dots\dots \\ a_{n1} & b_{n2} \dots b_{nn} \end{vmatrix} + (-1)^{n+1} \begin{vmatrix} a_{12} \dots a_{1n} & b_{12} \\ a_{22} \dots a_{2n} & b_{22} \\ \dots\dots\dots & \dots \\ a_{n2} \dots a_{nn} & b_{n2} \end{vmatrix} \begin{vmatrix} a_{11} & b_{11} & b_{13} \dots \\ a_{21} & b_{21} & b_{23} \dots \\ \dots & \dots & \dots\dots \\ a_{n1} & b_{n1} & b_{n3} \dots \end{vmatrix}$$

$$+ \dots\dots\dots\dots + (-1)^{2n-1} \begin{vmatrix} a_{12} \dots a_{1n} & b_{1n} \\ a_{22} \dots a_{2n} & b_{2n} \\ \dots\dots\dots & \dots \\ a_{n2} \dots a_{nn} & b_{nn} \end{vmatrix} \begin{vmatrix} a_{11} & b_{11} \dots b_{1,n-1} \\ a_{21} & b_{21} \dots b_{2,n-1} \\ \dots & \dots\dots\dots \\ a_{n1} & b_{n1} \dots b_{n,n-1} \end{vmatrix},$$

the index of the sign-factor of the first product being

$$(1+2+3+\ldots+n) + (2+3+\ldots+n+\overline{n+1}),$$

i.e.,
$$\tfrac{1}{2}n(n+1) + \tfrac{1}{2}n(n+1)+n,$$

for which n has been put, since $n(n+1)$ is even. Making the b column of each of the first factors pass over the columns before it, and making the a column of each of the second factors pass over to the place of the missing b column, we have

$$-\begin{vmatrix} b_{11} & a_{12} \ldots\ldots a_{1n} \\ b_{21} & a_{22} \ldots\ldots a_{2n} \\ \multicolumn{2}{c}{\ldots\ldots\ldots\ldots} \\ b_{n1} & a_{n2} \ldots\ldots a_{nn} \end{vmatrix} \begin{vmatrix} a_{11} & b_{12} \ldots\ldots b_{1n} \\ a_{21} & b_{22} \ldots\ldots b_{2n} \\ \multicolumn{2}{c}{\ldots\ldots\ldots\ldots} \\ a_{n1} & b_{n2} \ldots\ldots b_{nn} \end{vmatrix} - \begin{vmatrix} b_{12} & a_{12} \ldots\ldots a_{1n} \\ b_{22} & a_{22} \ldots\ldots a_{2n} \\ \multicolumn{2}{c}{\ldots\ldots\ldots\ldots} \\ b_{n2} & a_{n2} \ldots\ldots a_{nn} \end{vmatrix} \begin{vmatrix} b_{11} & a_{11} & b_{13} \ldots\ldots b_{1n} \\ b_{21} & a_{21} & b_{23} \ldots\ldots b_{2n} \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots} \\ b_{n1} & a_{n1} & b_{n3} \ldots\ldots b_{nn} \end{vmatrix}$$

$$- \ldots\ldots\ldots\ldots - \begin{vmatrix} b_{1n} & a_{12} \ldots\ldots a_{1n} \\ b_{2n} & a_{22} \ldots\ldots a_{2n} \\ \multicolumn{2}{c}{\ldots\ldots\ldots\ldots} \\ b_{nn} & a_{n2} \ldots\ldots a_{nn} \end{vmatrix} \begin{vmatrix} b_{11} \ldots\ldots b_{1,n-1} & a_{11} \\ b_{21} \ldots\ldots b_{2,n-1} & a_{21} \\ \multicolumn{2}{c}{\ldots\ldots\ldots\ldots\ldots} \\ b_{n1} \ldots\ldots b_{n,n-1} & a_{n1} \end{vmatrix},$$

the sign of every product being now $-$, since the number of changes of sign caused in the first is $n-1$, in the second n, in the third $n+1$, and so on.

Again, returning to Δ, and subtracting each element of the $(n+1)^{\text{th}}$ row from the corresponding element of the first row, each element of the $(n+2)^{\text{th}}$ row from the corresponding element of the second row, and so on, we have a determinant from whose first n rows it is possible to form but one non-zero minor of the n^{th} order, viz. $-|a_{1n}|$; hence it is seen (§ 79) that

$$\Delta = -|a_{1n}|\,|b_{1n}|.$$

This and the former expression obtained for Δ at once give the required identity in the special case under consideration.

Secondly, let the particular column fixed upon be not the first of $|a_{1n}|$, but some other, the k^{th} say. Then it is readily seen that to establish the theorem we have only to make this k^{th} column pass over the $k-1$ which precede it, apply to the product of the resulting determinant and $|b_{1n}|$ the already established case, and in every first factor of the result make the first column pass over the next $k-1$ columns.

Example. Taking the determinants $|a_1 b_2 c_3|$, $|x_1 y_2 z_3|$, and making each column in its turn the column for interchange, we have the six identities—

$$
\begin{aligned}
&\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \\
&= \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} a_1 & y_1 & z_1 \\ a_2 & y_2 & z_2 \\ a_3 & y_3 & z_3 \end{vmatrix} + \begin{vmatrix} y_1 & b_1 & c_1 \\ y_2 & b_2 & c_2 \\ y_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} x_1 & a_1 & z_1 \\ x_2 & a_2 & z_2 \\ x_3 & a_3 & z_3 \end{vmatrix} + \begin{vmatrix} z_1 & b_1 & c_1 \\ z_2 & b_2 & c_2 \\ z_3 & b_3 & z_3 \end{vmatrix}\begin{vmatrix} x_1 & y_1 & a_1 \\ x_2 & y_2 & a_2 \\ x_3 & y_3 & a_3 \end{vmatrix}, \\
&= |a_1 x_2 c_3|\,|b_1 y_2 z_3| + |a_1 y_2 c_3|\,|x_1 b_2 z_3| + |a_1 z_2 c_3|\,|x_1 y_2 b_3|, \\
&= |a_1 b_2 x_3|\,|c_1 y_2 z_3| + |a_1 b_2 y_3|\,|x_1 c_2 z_3| + |a_1 b_2 z_3|\,|x_1 y_2 c_3|, \\
&= |x_1 b_2 c_3|\,|a_1 y_2 z_3| + |a_1 x_2 c_3|\,|b_1 y_2 z_3| + |a_1 b_2 x_3|\,|c_1 y_2 z_3|, \\
&= |y_1 b_2 c_3|\,|x_1 a_2 z_3| + |a_1 y_2 c_3|\,|x_1 b_2 z_3| + |a_1 b_2 y_3|\,|x_1 c_2 z_3|, \\
&= |z_1 b_2 c_3|\,|x_1 y_2 a_3| + |a_1 z_2 c_3|\,|x_1 y_2 b_3| + |a_1 b_2 z_3|\,|x_1 y_2 c_3|.
\end{aligned}
$$

These are not all independent, any one being deducible from the other five by addition and subtraction. The number of different products they connect is seen to be ten.

§ 86. *The product of two determinants of the same order is equal to the sum of like products obtained from the original by interchanging* k *chosen columns of the one determinant with every set of* k *columns of the other in succession; the interchange of* k *columns with* k *columns being effected by interchanging the first column of the one set with the first column of the other, the second of the one with the second of the other, and so on.*

Let $|a_{1n}|$, $|b_{1n}|$ be the two determinants, and, first, let the k columns fixed upon be the first k columns of $|a_{1n}|$.

If a determinant of the $2n^{\text{th}}$ order be formed exactly as in § 85, and the first k columns of its minor $|a_{1n}|$ be then interchanged in order with the k columns of zeros below them, we have the determinant

$$\left|\begin{array}{cccccc} 0 \;\ldots\; 0 & a_{1,k+1} \;\ldots\; a_{1n} & b_{11} & b_{12} \;\ldots\; b_{1n} \\ 0 \;\ldots\; 0 & a_{2,k+1} \;\ldots\; a_{2n} & b_{21} & b_{22} \;\ldots\; b_{2n} \\ \cdots\cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \\ 0 \;\ldots\; 0 & a_{n,k+1} \;\ldots\; a_{nn} & b_{n1} & b_{n2} \;\ldots\; b_{nn} \\ a_{11} \;\ldots\; a_{1k} & 0 \;\ldots\; 0 & b_{11} & b_{12} \;\ldots\; b_{1n} \\ a_{21} \;\ldots\; a_{2k} & 0 \;\ldots\; 0 & b_{21} & b_{22} \;\ldots\; b_{2n} \\ \cdots\cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \\ a_{n1} \;\ldots\; a_{nk} & 0 \;\ldots\; 0 & b_{n1} & b_{n2} \;\ldots\; b_{nn} \end{array}\right| \quad \text{or} \quad \Delta, \text{ say}.$$

Preparing to express this as an aggregate of products of complementary minors of the n^{th} order formed from the first and last n rows, we see that the first factor of any non-zero product of this kind must contain the last $n-r$ columns of $|a_{1n}|$ and a set of r columns from $|b_{1n}|$, and that the cofactor must contain the remaining columns from $|a_{1n}|$ and $|b_{1n}|$; in other words, that this product is derivable from $|a_{1n}||b_{1n}|$ by transferring the first r columns of $|a_{1n}|$ to $|b_{1n}|$ and a set of r columns of $|b_{1n}|$ to $|a_{1n}|$,—the transferred columns occupying the last r places in the first factor and the first r places in the second. If the columns taken from $|b_{1n}|$ be the θ_1^{th}, θ_2^{th}, ..., θ_r^{th}, the index of the sign-factor of the product is

$$(1+2+\ldots+n)+(\overline{r+1}+\overline{r+2}+\ldots+n+\overline{n+\theta_1}+\overline{n+\theta_2}+\ldots+\overline{n+\theta_r});$$

$$\text{i.e.}\quad \tfrac{1}{2}n(n+1)+\left\{\tfrac{1}{2}(n-r)(n+r+1)+rn+\theta_1+\theta_2+\ldots+\theta_r\right\};$$

$$\text{i.e.}\quad \tfrac{1}{2}n(n+1)+\tfrac{1}{2}n(n+1)-\tfrac{1}{2}r(r+1)+rn+\theta_1+\theta_2+\ldots+\theta_r.$$

Now, were it not for the positions of the transferred columns, the series of products thus obtained would be that referred to in the statement of the theorem; and if, in each case, we pass the b columns of the first factor in order over the $n-r$ preceding a columns, and in the second factor transfer the r^{th} column to occupy the $\theta_r{}^{\text{th}}$ place, the $(r-1)^{\text{th}}$ column to occupy the $\theta_{r-1}{}^{\text{th}}$ place, and so on, we obtain the said series of products exactly. The number of changes of sign caused by these transferences of columns is seen to be

$$r(n-r)+(\overline{\theta_r-r}+\ldots+\overline{\theta_2-2}+\overline{\theta_1-1});$$

$$\text{i.e.,}\quad r(n-r)-\tfrac{1}{2}r(r+1)+\theta_1+\theta_2+\ldots+\theta_r.$$

Consequently the index of the sign-factor of the product will now be the sum of this number and the former index; so that, those parts of the sum being neglected which are even, the sign-factor is found to be

$$(-1)^{-r^2}\quad\text{i.e.}\quad(-1)^{r^2}\quad\text{i.e.}\quad(-1)^r.$$

Hence Δ is equal to the aggregate of the products referred to in the theorem each taken with the sign-factor $(-1)^r$.

Again, subtracting each element of the $(n+1)^{\text{th}}$ row of Δ from the corresponding element of the first row, each element of the $(n+2)^{\text{th}}$ row from the corresponding element of the second row, and so on, we have (§ 79) also

$$\Delta=(-1)^r\,|a_{1n}|\,|b_{1n}|.$$

Hence, by equating these two expressions for Δ, we have the required identity established for the particular case under consideration.

When the k columns fixed upon for interchange are not the first k columns of $|a_{1n}|$, we may prove the theorem by making them the first k columns and then using the already proved case, exactly after the manner of § 85.

EXAMPLE. Taking the determinants $|a_1b_2c_3d_4|$ and $|x_1y_2z_3w_4|$, and selecting the first two columns of the former as the columns for interchange, we have

$$\begin{aligned}|a_1b_2c_3d_4||x_1y_2z_3w_4| = {} & |x_1y_2c_3d_4||a_1b_2z_3w_4| + |x_1z_2c_3d_4||a_1y_2b_3w_4| + |x_1w_2c_3d_4||a_1y_2z_3b_4| \\ & + |y_1z_2c_3d_4||x_1a_2b_3w_4| + |y_1w_2c_3d_4||x_1a_2z_3b_4| + |z_1w_2c_3d_4||x_1y_2a_3b_4|.\end{aligned}$$

§ 87. From the elements of two determinants $|a_{1n}|$, $|b_{1n}|$ a number of different zero-determinants of the $2n^{\text{th}}$ order can be formed resembling Δ in § 86, and thus a corresponding number of identities similar to the one there established can be at once obtained by expanding in terms of products of complementary minors of the n^{th} order. Any one of the first n columns of such a zero-determinant is formed by taking for the one half of it the corresponding column of $|a_{1n}|$ and for the other either a repetition of this or n zeros; the last n columns are formed in like manner, but from $|b_{1n}|$; and the number of columns independent of zeros is not less than $n+1$, this being necessary in order that the determinant may vanish in accordance with the theorem of § 80. If, in the determinant, $|a_{1n}|$ and $|b_{1n}|$ are complementary minors, the resulting identity may be expressed like that of § 86, namely, so as to give an equivalent for the product of $|a_{1n}|$ and $|b_{1n}|$.

EXAMPLE. Taking the determinants $|a_1b_2c_3|$, $|x_1y_2z_3|$, and seeking to express their product as an aggregate of like products, in which the first factor shall contain the column of b's and the second factor the column of x's, we form the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 & y_1 & z_1 \\ a_2 & b_2 & c_2 & 0 & y_2 & z_2 \\ a_3 & b_3 & c_3 & 0 & y_3 & z_3 \\ a_1 & 0 & c_1 & x_1 & y_1 & z_1 \\ a_2 & 0 & c_2 & x_2 & y_2 & z_2 \\ a_3 & 0 & c_3 & x_3 & y_3 & z_3 \end{vmatrix},$$

which (§ 78) is equal to

$$|a_1 b_2 c_3||x_1 y_2 z_3| + |a_1 b_2 y_3||c_1 x_2 z_3| - |a_1 b_2 z_3||c_1 x_2 y_3| + |b_1 c_2 y_3||a_1 x_2 z_3| - |b_1 c_2 z_3||a_1 x_2 y_3|$$
$$- |b_1 y_2 z_3||a_1 c_2 x_3|.$$

But by subtracting each element of the last three rows from the corresponding element of the first three rows, it is at once seen to be also equal to 0; hence

$$\begin{aligned} &|a_1 b_2 c_3||x_1 y_2 z_3| \\ = {} & |a_1 b_2 y_3||x_1 c_2 z_3| + |a_1 b_2 z_3||x_1 y_2 c_3| + |y_1 b_2 c_3||x_1 a_2 z_3| + |z_1 b_2 c_3||x_1 y_2 a_3| - |y_1 b_2 z_3||x_1 a_2 c_3|. \end{aligned}$$

§ 88. If the two given determinants in the immediately preceding theorems have one or more columns in common, the number of products in the resulting identity is less (§ 27) than it would otherwise be. Special cases of this kind are of sufficiently frequent occurrence to merit the student's attention.

§ 89. It is readily seen that identities, similar to those of § 87, but having for the factors of each product two determinants of *different* orders, might also be established. They do not admit of very simple statement, but are often of use.

EXAMPLE. Since

$$\begin{vmatrix} b_1 & b_2 & 0 & b_3 & b_4 \\ c_1 & c_2 & 0 & c_3 & c_4 \\ b_1 & 0 & b_2 & b_3 & b_4 \\ c_1 & 0 & c_2 & c_3 & c_4 \\ d_1 & 0 & d_2 & d_3 & d_4 \end{vmatrix} = 0,$$

it at once follows that

$$|b_1 c_2||b_2 c_3 d_4| = |b_2 c_3||b_1 c_2 d_4| - |b_2 c_4||b_1 c_2 d_3|.$$

§ 90. *The product of a determinant and any one of its minors* $\mathfrak{m}$ *is expressible as an aggregate of products of pairs of minors: the first factors of the products being obtained by taking* q *rows in which the rows of* $\mathfrak{m}$ *are included and* *forming from them every minor of the* q^th^ *order which contains* $\mathfrak{m}$*; the second factor of any product being that minor which includes* $\mathfrak{m}$ *and the complementary of the first factor; and the sign of any product being fixed by transforming the second factor so as to have its principal diagonal coincident with those of the two minors which it was formed to include, and then taking + or − according as the sum of the numbers indicating the rows and columns from which the first factor was formed is even or odd.*

Let $|a_{1n}|$ be the given determinant, $|a_{pp}a_{p+1,p+1}\dots a_{qq}|$ the chosen minor $\mathfrak{m}$, and let the q rows of $|a_{1n}|$ taken to include the rows of $\mathfrak{m}$ be the *first* q rows.

We have at once an equivalent for the product of $|a_{1n}|$ and its minor $\mathfrak{m}$, by forming a new determinant in which $|a_{1n}|$ and $\mathfrak{m}$ are complementary minors, the former being situated in the first n rows and first n columns with nothing but zero-elements below it. Such a determinant is

$$\left|\begin{array}{cccccccccccccc}
a_{11} & a_{12} & \dots & a_{1,p-1} & a_{1p} & \dots & a_{1q} & a_{1,q+1} & \dots & a_{1n} & 0 & \dots & 0 \\
a_{21} & a_{22} & \dots & a_{2,p-1} & a_{2p} & \dots & a_{2q} & a_{2,q+1} & \dots & a_{2n} & 0 & \dots & 0 \\
\multicolumn{13}{c}{\dotfill} \\
a_{p-1,1} & a_{p-1,2} & \dots & a_{p-1,p-1} & a_{p-1,p} & \dots & a_{p-1,q} & a_{p-1,q+1} & \dots & a_{p-1,n} & 0 & \dots & 0 \\
a_{p1} & a_{p2} & \dots & a_{p,p-1} & a_{pp} & \dots & a_{pq} & a_{p,q+1} & \dots & a_{p,n} & 0 & \dots & 0 \\
\multicolumn{13}{c}{\dotfill} \\
a_{q1} & a_{q2} & \dots & a_{q,p-1} & a_{qp} & \dots & a_{qq} & a_{q,q+1} & \dots & a_{q,n} & 0 & \dots & 0 \\
a_{q+1,1} & a_{q+1,2} & \dots & a_{q+1,p-1} & a_{q+1,p} & \dots & a_{q+1,q} & a_{q+1,q+1} & \dots & a_{q+1,n} & a_{q+1,p} & \dots & a_{q+1,q} \\
\multicolumn{13}{c}{\dotfill} \\
a_{n1} & a_{n2} & \dots & a_{n,p-1} & a_{np} & \dots & a_{nq} & a_{n,q+1} & \dots & a_{nn} & a_{np} & \dots & a_{nq} \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & a_{pp} & \dots & a_{pq} \\
\multicolumn{13}{c}{\dotfill} \\
0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & a_{qp} & \dots & a_{qq}
\end{array}\right| \text{ or } \Delta, \text{ say,}$$

where it has to be specially noticed that the q chosen rows of $|a_{1n}|$ are prolonged with zeros, and that each of the other rows is prolonged by repeating in order the elements of it which are in the same column with any of the elements of $\mathfrak{m}$.

In Δ the minor $\mathfrak{m}$ occurs twice. Adding each element of the first set of rows to which $\mathfrak{m}$ belongs to the corresponding element of the second set, and then subtracting each element of the second set of columns to which $\mathfrak{m}$ belongs from the corresponding element of the first set, we find

$$\Delta = \left|\begin{array}{ccccccccccccc}
a_{11} & a_{12} & \ldots & a_{1,p-1} & a_{1p} & \ldots & a_{1q} & a_{1,q+1} & \ldots & a_{1n} & 0 & \ldots & 0 \\
a_{21} & a_{22} & \ldots & a_{2,p-1} & a_{2p} & \ldots & a_{2q} & a_{2,q+1} & \ldots & a_{2n} & 0 & \ldots & 0 \\
\multicolumn{13}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
a_{p-1,1} & a_{p-1,2} & .. & a_{p-1,p-1} & a_{p-1,p} & .. & a_{p-1,q} & a_{p-1,q+1} & .. & a_{p-1,n} & 0 & \ldots & 0 \\
a_{p1} & a_{p2} & \ldots & a_{p,p-1} & a_{pp} & \ldots & a_{pq} & a_{p,q+1} & \ldots & a_{pn} & 0 & \ldots & 0 \\
\multicolumn{13}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
a_{q1} & a_{q2} & \ldots & a_{q,p-1} & a_{qp} & \ldots & a_{qq} & a_{q,q+1} & \ldots & a_{qn} & 0 & \ldots & 0 \\
a_{q+1,1} & a_{q+1,2} & .. & a_{q+1,p-1} & 0 & \ldots & 0 & a_{q+1,q+1} & .. & a_{q+1,n} & a_{q+1,p} & .. & a_{q+1,q} \\
\multicolumn{13}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
a_{n1} & a_{n2} & \ldots & a_{n,p-1} & 0 & \ldots & 0 & a_{n,q+1} & \ldots & a_{nn} & a_{np} & \ldots & a_{nq} \\
a_{p1} & a_{p2} & \ldots & a_{p,p-1} & 0 & \ldots & 0 & a_{p,q+1} & \ldots & a_{pn} & a_{pp} & \ldots & a_{pq} \\
\multicolumn{13}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
a_{q1} & a_{q2} & \ldots & a_{q,p-1} & 0 & \ldots & 0 & a_{q,q+1} & \ldots & a_{qn} & a_{qp} & \ldots & a_{qq}
\end{array}\right|.$$

If now we take the first q rows of this determinant, and form every minor of the q^{th} order preparatory to finding the expansion of the determinant as an aggregate of products of complementary minors, we see that, although the full list of minors would be exactly the same as if we had been dealing with $|a_{1n}|$, we need only take those which include the selected minor $\mathfrak{m}$, because all the others have here complementaries which vanish; also, we see that each of the complementaries of those thus taken includes the complementary of the same minor in $|a_{1n}|$ and the selected minor besides, and that each is itself a minor of $|a_{1n}|$, being formed from those $n-p+1$ rows of $|a_{1n}|$ which are made up of the $n-q$ rows not included in the chosen q rows and the $q-p+1$ rows in which $\mathfrak{m}$ is situated.

But this aggregate of products is exactly the aggregate of products specified in the enunciation of the theorem, and as it is the equivalent of Δ and therefore of $|a_{1n}| \times \mathfrak{m}$, the theorem has been established.

Examples:—Taking the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}$$

and its minor

$$\begin{vmatrix} b_3 & b_4 \\ c_3 & c_4 \end{vmatrix},$$

we have

$$\begin{aligned} |b_3c_4||a_1b_2c_3d_4e_5| &= |a_1b_2c_3d_4||b_3c_4e_5| - |a_1b_3c_4d_5||b_3c_4e_2| + |a_2b_3c_4d_5||b_3c_4e_1|, \\ &= -|a_1b_2c_3c_4||b_3c_4d_5| + |a_1b_3c_4e_5||b_3c_4d_2| - |a_2b_3c_4e_5||b_3c_4d_1|, \\ &= \ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

the factors of the fourth order being formed in the first case from the first, second, third, fourth rows, and in the second from the first, second, third, fifth rows.

Taking a minor of the next lower order for the co-factor of $|a_1b_2c_3d_4e_5|$, we have

$$\begin{aligned} d_2|a_1b_2c_3d_4e_5| &= \left.\begin{aligned} &|c_1d_2e_3||a_4b_5d_2| - |c_1d_2e_4||a_3b_5d_2| + |c_1d_2e_5||a_3b_4d_2| \\ &+ |c_2d_3e_4||a_1b_5d_2| - |c_2d_3e_5||a_1b_4d_2| + |c_2d_4e_5||a_1b_3d_2| \end{aligned}\right\}, \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &= \left.\begin{aligned} -|d_1e_2||a_3b_4c_5d_2| + |d_2e_3||a_1b_4c_5d_2| - |d_2e_4||a_1b_3c_5d_2| \\ + |d_2e_5||a_1b_3c_4d_2| \end{aligned}\right\}, \\ &= \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

If the co-factor of $|a_1b_2c_3d_4e_5|$ be of a lower order still, namely, the order 0, and as such be taken equal to unity, we have the theorem of § 77, which in this way we may view as being here generalized.

§ 91. At the opposite extreme from the theorem of § 77 viewed as a case of the foregoing, we have another theorem of sufficient importance to be specially noticed. This is the case in which, the original determinant being of the n^{th} order, its co-factor $\mathfrak{m}$ is of the $(n-2)^{\text{th}}$. Here the rows from which the first factors of the development are formed must be $n-1$ in number, and as the said first factors must include $\mathfrak{m}$, there can be only two of them, so that the development must consist of two terms which are each the product of two determinants of the $(n-1)^{\text{th}}$ order.

Examples:—

$$|b_2c_3d_4||a_1b_2c_3d_4e_5| = |b_2c_3d_4e_5||a_1b_2c_3d_4| - |b_1c_2d_3e_4||a_2b_3c_4d_5|;$$
$$|a_1b_2c_3||a_1b_2c_3d_4e_5| = |a_1b_2c_3d_4||a_1b_2c_3e_5| - |a_1b_2c_3d_5||a_1b_2c_3e_4|.$$

Denoting $|a_1b_2c_3d_4e_5|$ by D, we may (§ 60) write the first of these in the form

$$D\frac{\partial^2 D}{\partial a_1 \partial e_5} = \frac{\partial D}{\partial a_1}\frac{\partial D}{\partial e_5} - \frac{\partial D}{\partial a_5}\frac{\partial D}{\partial e_1};$$

similarly, the second; and, quite generally, we have

$$D(a_{1n})\frac{\partial^2 D}{\partial a_{hr} \partial a_{ks}} = \frac{\partial D}{\partial a_{hr}}\frac{\partial D}{\partial a_{ks}} - \frac{\partial D}{\partial a_{hs}}\frac{\partial D}{\partial a_{kr}},$$

—the form in which the theorem is commonly quoted.

§ 92. The theorems of §§ 77, 90 are connected in another way, which it is of still greater importance to observe. As an instance of the latter theorem we have (p. 131)

$$\begin{aligned}&|b_3c_4||a_1b_2c_3d_4e_5| \\ &\quad = |a_1b_2c_3d_4||b_3c_4e_5| - |a_1b_3c_4d_5||b_3c_4e_2| + |a_2b_3c_4d_5||b_3c_4e_1|.\end{aligned}$$

If now, in place of each determinant here, we substitute the co-factor which it has in $|a_1b_2c_3d_4e_5|$, we obtain the statement

$$|a_1d_2e_5| = e_5|a_1d_2| - e_2|a_1d_5| + e_1|a_2d_5|,$$

which is at once recognised as an instance of the theorem of § 77. The identities

$$\left.\begin{aligned} d_2|a_1b_2c_3d_4e_5| = {} & |c_1d_2e_3||a_4b_5d_2| - |c_1d_2e_4||a_3b_5d_2| + |c_1d_2e_5||a_3b_4d_2| \\ & + |c_2d_3e_4||a_1b_5d_2| - |c_2d_3e_5||a_1b_4d_2| + |c_2d_4e_5||a_1b_3d_2| \end{aligned}\right\},$$

$$\left.\begin{aligned} |a_1b_3c_4e_5| = {} & |a_4b_5||c_1e_3| - |a_3b_5||c_1e_4| + |a_3b_4||c_1e_5| \\ & + |a_1b_5||c_3e_4| - |a_1b_4||c_3e_5| + |a_1b_3||c_4e_5| \end{aligned}\right\},$$

are similarly related; and generally it is found that to every instance of the one theorem there corresponds in this way an instance of the other, so that the two may be spoken of as *complementary theorems.*

§ 93. *If the first* k *elements in the first* k − 1 *rows of a determinant* Δ_n *be taken with the first* k *elements of the other rows in succession to form as determinants of the* k*th order the elements of the first column of a new determinant* Δ_{n-k+1}, *and if the determinants formed in like manner from the second* k *consecutive columns of* Δ_n *be made the elements of the second column of* Δ_{n-k+1}, *and so on, then* Δ_n *is equal to* Δ_{n-k+1} *divided by the product of all the determinants of the* (k − 1)*th order, except the first and last, formed from the first* k − 1 *rows and any* k − 1 *consecutive columns of* Δ_n.

The case of this in which $k=2$ has been already established (§ 53); so that if $\Delta_n = |a_{1n}|$, we know that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \dots\dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots\dots a_{2n} \\ a_{31} & a_{32} & a_{33} \dots\dots a_{3n} \\ \dots\dots & \dots\dots & \dots\dots\dots\dots \\ a_{n1} & a_{n2} & a_{n3} \dots\dots a_{nn} \end{vmatrix} = \frac{1}{a_{12}a_{13} \dots\dots a_{1,n-1}} \begin{Vmatrix} |a_{11}a_{22}| & |a_{12}a_{23}| & \dots & |a_{1,n-1}\,a_{2n}| \\ |a_{11}a_{32}| & |a_{12}a_{33}| & \dots & |a_{1,n-1}\,a_{3n}| \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ |a_{11}a_{n2}| & |a_{11}a_{n3}| & \dots & |a_{1,n-1}a_{nn}| \end{Vmatrix}. \quad (1)$$

Transforming the second determinant here by means of the identity in which it occurs, and putting

$$\begin{vmatrix} |a_{11}a_{22}| & |a_{12}a_{23}| \\ |a_{11}a_{32}| & |a_{12}a_{33}| \end{vmatrix} = a_{12}|a_{11}a_{22}a_{33}|,$$

$$\begin{vmatrix} |a_{12}a_{23}| & |a_{13}a_{24}| \\ |a_{12}a_{33}| & |a_{13}a_{34}| \end{vmatrix} = a_{13}|a_{12}a_{23}a_{34}|,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

as § 91 entitles us to do, we obtain the result

$$\frac{\begin{vmatrix} a_{12}|a_{11}a_{22}a_{33}| & a_{13}|a_{12}a_{23}a_{34}| & \ldots & a_{1,n-1}|a_{1,n-2}a_{2,n-1}a_{3n}| \\ a_{12}|a_{11}a_{22}a_{43}| & a_{13}|a_{12}a_{23}a_{44}| & \ldots & a_{1,n-1}|a_{1,n-2}a_{2,n-1}a_{4n}| \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ a_{12}|a_{11}a_{22}a_{n3}| & a_{13}|a_{12}a_{23}a_{n4}| & \ldots & a_{1,n-1}|a_{1,n-2}a_{2,n-1}a_{nn}| \end{vmatrix}}{|a_{12}a_{23}|\,|a_{13}a_{24}|\ldots.|a_{1,n-2}a_{2,n-1}|},$$

so that by substitution in (1) we have

$$|a_{1n}| = \frac{\begin{vmatrix} |a_{11}a_{22}a_{33}| & |a_{12}a_{23}a_{34}| & \ldots & |a_{1,n-2}a_{2,n-1}a_{3n}| \\ |a_{11}a_{22}a_{43}| & |a_{12}a_{23}a_{44}| & \ldots & |a_{1,n-2}a_{2,n-1}a_{4n}| \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ |a_{11}a_{22}a_{n3}| & |a_{12}a_{23}a_{n4}| & \ldots & |a_{1,n-2}a_{2,n-1}a_{nn}| \end{vmatrix}}{|a_{12}a_{23}|\,|a_{13}a_{24}|\ldots.|a_{1,n-2}a_{2,n-1}|}, \quad \ldots(2)$$

which is the next case of the theorem.

In exactly similar fashion it follows from this that

$$|a_{1n}| = \frac{\begin{vmatrix} |a_{11}a_{22}a_{33}a_{44}| & \ldots & |a_{1,n-3}a_{2,n-2}a_{3,n-1}a_{4n}| \\ |a_{11}a_{22}a_{33}a_{54}| & \ldots & |a_{1,n-3}a_{2,n-2}a_{3,n-1}a_{5n}| \\ \cdots\cdots & \cdots & \cdots\cdots \\ |a_{11}a_{22}a_{33}a_{n4}| & \ldots & |a_{1,n-3}a_{2,n-2}a_{3,n-1}a_{nn}| \end{vmatrix}}{|a_{12}a_{23}a_{34}|\ldots.|a_{1,n-3}a_{2,n-2}a_{3,n-1}|}, \quad \ldots(3)$$

and so on; the extreme case being the identity repeatedly used in the demonstration, viz., that of § 91.

Similarly an extension of the identity of Ex. 20, Set X., might be established, giving

$$|a_{1n}| = \begin{vmatrix} |a_{11}a_{22}a_{33}| & |a_{11}a_{22}a_{34}| & \dots\dots & |a_{11}a_{22}a_{3n}| \\ |a_{11}a_{22}a_{43}| & |a_{11}a_{22}a_{44}| & \dots\dots & |a_{11}a_{22}a_{4n}| \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ |a_{11}a_{22}a_{n3}| & |a_{11}a_{22}a_{n4}| & \dots\dots & |a_{11}a_{22}a_{nn}| \end{vmatrix} \div |a_{11}a_{22}|^{n-3},$$

and so forth.

§ 94. The determinant each of whose elements is the cofactor of the corresponding element in another determinant is called the determinant *adjugate* to that other. Thus

$$\begin{vmatrix} |b_2c_3| & -|a_2c_3| & |a_2b_3| \\ -|b_1c_3| & |a_1c_3| & -|a_1b_3| \\ |b_1c_2| & -|a_1c_2| & |a_1b_2| \end{vmatrix}$$

is the determinate adjugate to

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

and using the notation of § 49,

$$|A_{11}A_{22}\dots\dots A_{nn}| \quad \text{or} \quad |A_{1n}| \quad \text{or} \quad D(A_{1n})$$

is the determinant adjugate to

$$|a_{11}a_{22}\dots\dots a_{nn}| \quad \text{or} \quad |a_{1n}| \quad \text{or} \quad D(a_{1n}).$$

When the elements of the adjugate determinant are specified as above by means of the complementary minors of the corresponding elements in the original determinant, negative signs must appear (§ 59) in the places whose row-number and column-number have a sum which is odd.

These signs may however (Ex. 22, Set VII.) be deleted without altering the value of the determinant; hence, in the definition which has been given we might substitute "complementary minor" for "co-factor."

§ 95. *The determinant adjugate to a determinant of the* n*th* *degree is equal to the* (n−1)*th* *power of the latter.*

Let the given determinant be

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots\ldots a_{2n} \\ a_{31} & a_{32} & a_{33} \ldots\ldots a_{3n} \\ \cdots & \cdots & \cdots\cdots\cdots\cdots \\ a_{n1} & a_{n2} & a_{n3} \ldots\ldots a_{nn} \end{vmatrix}, \qquad \text{or} \qquad |a_{1n}|.$$

Multiplying it by its adjugate

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \ldots\ldots A_{1n} \\ A_{21} & A_{22} & A_{23} \ldots\ldots A_{2n} \\ A_{31} & A_{32} & A_{33} \ldots\ldots A_{3n} \\ \cdots & \cdots & \cdots\cdots\cdots\cdots \\ A_{n1} & A_{n2} & A_{n3} \ldots\ldots A_{nn} \end{vmatrix}, \qquad \text{or} \qquad A_{1n}|,$$

after the manner of § 67, the first column of the new determinant is

$$\begin{array}{l} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + \ldots\ldots + a_{1n}A_{1n} \\ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} + \ldots\ldots + a_{2n}A_{1n} \\ a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} + \ldots\ldots + a_{3n}A_{1n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}A_{11} + a_{n2}A_{12} + a_{n3}A_{13} + \ldots\ldots + a_{nn}A_{1n} \end{array}$$

the first expression of which is (§ 46) equal to $|a_{1n}|$, and each of the others (§ 52) equal to zero.

In like manner the elements of the second column are all zero, except the second element, which is $|a_{1n}|$; and so on. Thus

$$|a_{1n}| \times |A_{1n}| = \begin{vmatrix} |a_{1n}| & 0 & 0 & \dots & 0 \\ 0 & |a_{1n}| & 0 & \dots & 0 \\ 0 & 0 & |a_{1n}| & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & |a_{1n}| \end{vmatrix},$$

$$= |a_{1n}|^n;$$

$$\therefore\quad |A_{1n}| = |a_{1n}|^{n-1},$$

as was to be proved.

§ 96. *If the determinant adjugate to a given determinant be formed, any minor of it of the* m^th^ *order is equal to the product obtained by multiplying the co-factor of the corresponding minor in the original determinant by the* (m−1)*th power of the original determinant.*

I. When the minor of the adjugate is $|A_{11}A_{22}\dots A_{mm}|$, the original determinant being

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} & a_{1,m+1} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} & a_{2,m+1} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3m} & a_{3,m+1} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mm} & a_{m,m+1} & \dots & a_{mn} \\ a_{m+1,1} & a_{m+1,2} & a_{m+1,3} & \dots & a_{m+1,m} & a_{m+1,m+1} & \dots & a_{m+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} & a_{n,m+1} & \dots & a_{nn} \end{vmatrix}.$$

Taking the adjugate determinant and changing the elements of all the rows after the m^{th}, those occupying the principal diagonal into 1, and all the others into 0, we have

$$\left|\begin{array}{cccccccc}
A_{11} & A_{12} & \cdots\cdots & A_{1m} & A_{1,m+1} & A_{1,m+2} & \cdots\cdots & A_{1,n} \\
A_{21} & A_{22} & \cdots\cdots & A_{2m} & A_{2,m+1} & A_{2,m+2} & \cdots\cdots & A_{2,n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
A_{m,1} & A_{m,2} & \cdots\cdots & A_{m,m} & A_{m,m+1} & A_{m,m+2} & \cdots\cdots & A_{m,n} \\
0 & 0 & \cdots\cdots & 0 & 1 & 0 & \cdots\cdots & 0 \\
0 & 0 & \cdots\cdots & 0 & 0 & 1 & \cdots\cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots\cdots & 0 & 0 & 0 & \cdots\cdots & 1
\end{array}\right|,$$

which is clearly equal to the chosen minor $|A_{11} \cdots\cdots A_{mm}|$. Multiplying the original determinant by this, there results

$$|a_{1n}| \times |A_{1m}| = \left|\begin{array}{ccccccccc}
|a_{1n}| & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & |a_{1n}| & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & |a_{1n}| & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & |a_{1n}| & 0 & \cdots & 0 \\
a_{1,m+1} & a_{2,m+1} & a_{3,m+1} & \cdots & a_{m,m+1} & a_{m+1,m+1} & \cdots & a_{n,m+1} \\
a_{1,m+2} & a_{2,m+2} & a_{3,m+2} & \cdots & a_{m,m+2} & a_{m+1,m+2} & \cdots & a_{n,m+2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{1,n} & a_{2,n} & a_{3,n} & \cdots & a_{m,n} & a_{m+1,n} & \cdots & a_{nn}
\end{array}\right|,$$

$$= |a_{1n}|^m \quad |a_{m+1,m+1} \cdots a_{nn}|;$$

$$\therefore \quad |A_{1m}| = |a_{1n}|^{m-1}|a_{m+1,m+1} \cdots a_{nn}|;$$

and $|a_{m+1,m+1} \cdots\cdots a_{nn}|$ being, in the original determinant, the co-factor of the minor corresponding to the chosen minor, one case of the theorem is established.

II. When the minor of the adjugate is any other than $|A_{11}A_{22} \cdots A_{mm}|$.

Let the rows from which the elements of the minor are taken be the h^{th}, k^{th}, l^{th}, ..., and the columns the r^{th}, s^{th}, u^{th}, ..., so that the minor may be denoted by $|A_{hr}A_{ks}A_{lu}\ldots.|$, and let $h+k+l+\ldots.+r+s+u+\ldots.=\sigma$.

Translating the said rows in order upwards and the said columns in order towards the left, the chosen minor will occupy the place at first occupied by $|A_{11}A_{22}\ldots.A_{mm}|$; and if we change the elements in the rows after the m^{th}, making those 1 which occupy places in the principal diagonal and all the others 0, we have as before a determinant of the n^{th} order, which equals the chosen minor

$$|A_{hr}A_{ks}A_{lu}\ldots.|.$$

Also, translating in the same way the corresponding rows and columns in the original determinant, we have a determinant which (see § 75) is equal to

$$(-1)^{\sigma}|a_{1n}|.$$

In the former of these two resulting determinants each A of the first m rows occupies the place which the corresponding a occupies in the latter determinant; consequently on multiplying the two together we have as before

$$|a_{1n}|^{m} \times \text{ co-factor of } |a_{hr}a_{ks}a_{lu}\ldots.| \text{ in } (-1)^{\sigma}|a_{1n}|,$$

and on division by $(-1)^{\sigma}|a_{1n}|$ there results

$$|A_{hr}A_{ks}A_{lu}\ldots.| = |a_{1n}|^{m-1} \times \text{ co-factor of } |a_{hr}a_{ks}\ldots.| \text{ in } |a_{1n}|,$$

as was to be proved.

If we write A_{hr} for the complementary minor of a_{hr} in $|a_{1n}|$, then since $A_{hr}=(-1)^{h+r}\,\mathrm{A}_{hr}$, &c., and the co-factor of $|a_{hr}a_{ks}a_{lu}\ldots|$ in $|a_{1n}|$ is equal to $(-1)^{\sigma}$ multiplied by the complementary minor of $|a_{hr}a_{ks}a_{lu}\ldots|$ in $|a_{1n}|$, the result just obtained becomes

$$|(-1)^{h+r}\mathrm{A}_{hr}\,(-1)^{k+s}\mathrm{A}_{ks}\ldots.| = |a_{1n}|^{m-1} \times (-1)^{\sigma} \times \text{ complementary of } |a_{hr}\,a_{ks}\ldots.|;$$

so that, multiplying the rows of the left-hand member of this by $(-1)^h$, $(-1)^k$, ... respectively, and the columns by $(-1)^r$, $(-1)^s$, ... respectively, and multiplying the right-hand member by the same, viz. by $(-1)^\sigma$, we have

$$|A_{hr} A_{ks} A_{lu} \ldots\ldots| = |a_{1n}|^{m-1} \times \text{complementary of } |a_{hr} a_{ks} a_{lu} \ldots\ldots|.$$

Example. The adjugate of

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

is

$$\begin{vmatrix} |b_2 c_3 d_4| & -|b_1 c_3 d_4| & |b_1 c_2 d_4| & -|b_1 c_2 d_3| \\ -|a_2 c_3 d_4| & |a_1 c_3 d_4| & -|a_1 c_2 d_4| & |a_1 c_2 d_3| \\ |a_2 b_3 d_4| & -|a_1 b_3 d_4| & |a_1 b_2 d_4| & -|a_1 b_2 d_3| \\ -|a_2 b_3 c_4| & |a_1 b_3 c_4| & -|a_1 b_2 c_4| & |a_1 b_2 c_3| \end{vmatrix},$$

and if the chosen minor be

$$\begin{vmatrix} |a_1 c_3 d_4| & -|a_1 c_2 d_4| \\ |a_1 b_3 c_4| & -|a_1 b_2 c_4| \end{vmatrix},$$

we take the adjugate and by transposition of rows and columns obtain

$$\begin{vmatrix} |a_1 c_3 d_4| & -|a_1 c_2 d_4| & -|a_2 c_3 d_4| & |a_1 c_2 d_3| \\ |a_1 b_3 c_4| & -|a_1 b_2 c_4| & -|a_2 b_3 c_4| & |a_1 b_2 c_3| \\ -|b_1 c_3 d_4| & |b_1 c_2 d_4| & |b_2 c_3 d_4| & -|b_1 c_2 d_3| \\ -|a_1 b_3 d_4| & |a_1 b_2 d_4| & |a_2 b_3 d_4| & -|a_1 b_2 d_3| \end{vmatrix};$$

then altering the elements of the last two rows we have

$$\begin{vmatrix} |a_1 c_3 d_4| & -|a_1 c_2 d_4| & -|a_2 c_3 d_4| & |a_1 c_2 d_3| \\ |a_1 b_3 c_4| & -|a_1 b_2 c_4| & -|a_2 b_3 c_4| & |a_1 b_2 c_3| \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Multiplying this by

$$\begin{vmatrix} b_2 & b_3 & b_1 & b_4 \\ d_2 & d_3 & d_1 & d_4 \\ a_2 & a_3 & a_1 & a_4 \\ c_2 & c_3 & c_1 & c_4 \end{vmatrix}$$

there results

$$|b_2d_3a_1c_4|\begin{vmatrix}|a_1c_3d_4| & -|a_1c_2d_4| \\ |a_1b_3c_4| & -|a_1b_2c_4|\end{vmatrix} = \begin{vmatrix}|a_1b_2c_2d_4| & 0 & 0 & 0 \\ 0 & |a_1b_2c_3d_4| & 0 & 0 \\ b_1 & d_1 & a_1 & c_1 \\ b_4 & d_4 & a_4 & c_4\end{vmatrix},$$

$$= |a_1b_2c_3d_4|^2\ |a_1c_4|;$$

$$\therefore\ \begin{vmatrix}|a_1c_3d_4| & |a_1c_2d_4| \\ |a_1b_3c_4| & |a_1b_2c_4|\end{vmatrix} = |a_1b_2c_3d_4|\ |a_1c_4|.$$

This example serves also to illustrate the fact that the special case of the theorem of § 90, noticed in § 91, is at the same time a special case ($m=2$) of the present theorem.

§ 97. It immediately follows from the preceding that, in the case of a determinant which is equal to zero, all the minors of the adjugate which are of a higher order than the first must also be equal to zero. Thus, taking minors of the second order,

$$\text{if } |a_{1n}| = 0,\quad \begin{vmatrix}A_{h1} & A_{h2} \\ A_{k1} & A_{k2}\end{vmatrix} = \begin{vmatrix}A_{h1} & A_{h3} \\ A_{k1} & A_{k3}\end{vmatrix} = \ldots\ldots = 0,$$

$$\text{and } \therefore\ A_{h1} : A_{k1} :: A_{h2} : A_{k2} :: A_{h3} : A_{k3} :: \ldots\ldots$$

$$\text{or}\quad A_{h1} : A_{h2} : A_{h3} : \ldots\ldots :: A_{k1} : A_{k2} : A_{k3} : \ldots\ldots$$

that is to say, in the case of a zero-determinant the co-factors of the elements of any one row are in order proportional to the co-factors of the elements of any other row.

§ 98. *To every general theorem which takes the form of an identical relation between a number of the minors of a determinant or between the determinant itself and a number of its minors, there corresponds another theorem derivable from the former by merely substituting for every minor its co-factor in the determinant, and then multiplying any* *term by such a power of the determinant as will make all the terms of the same degree.*

Let the established identity in regard to $|a_{1n}|$ be

$$\mathrm{M}_r\mathrm{M}_s\mathrm{M}_t\ldots + \ldots\ldots = \mathrm{M}_\rho\mathrm{M}_\sigma\mathrm{M}_\tau\ldots + \ldots\ldots$$

where M_r is used to denote some minor of $|a_{1n}|$ of the order r, and where consequently $r+s+t+\ldots\ldots = \rho+\sigma+\tau+\ldots\ldots$

Now since the identity holds in regard to every determinant, it holds in regard to $|A_{1n}|$ the adjugate of $|a_{1n}|$, hence if $\mathfrak{m}$ stand for the minor of $|A_{1n}|$ corresponding to the minor M_r of $|a_{1n}|$, it follows that

$$\mathfrak{m}_r\mathfrak{m}_s\mathfrak{m}_t\ldots + \ldots\ldots = \mathfrak{m}_\rho\mathfrak{m}_\sigma\mathfrak{m}_\tau\ldots + \ldots\ldots$$

Substituting for every $\mathfrak{m}$ its equivalent as given by the theorem of § 96, and, in order to do so, denoting the cofactor of M_r in $|a_{1n}|$ by M'_{n-r} we have

$$\begin{aligned}|a_{1n}|^{r-1}\mathrm{M}'_{n-r}\,.\,|a_{1n}|^{s-1}\mathrm{M}'_{n-s}\,.\,|a_{1n}|^{t-1}\mathrm{M}'_{n-t}\ldots + \ldots\ldots\\ = |a_{1n}|^{\rho-1}\mathrm{M}'_{n-\rho}\,.\,|a_{1n}|^{\sigma-1}\mathrm{M}'_{n-\sigma}\,.\,|a_{1n}|^{\tau-1}\mathrm{M}'_{n-\tau}\ldots + \ldots\ldots\end{aligned}$$

whence, on division by the lowest power of $|a_{1n}|$ contained in any term, there results the identity which was to be established.

This is the *Law of Complementaries* incidentally exemplified in § 92.

§ 99. By the application of the Law of Complementaries, some of the already established theorems furnish new theorems of considerable interest. As an example the identity of § 53 may be taken, a particular case of which is

$$|a_1b_2c_3d_4|b_1c_1 = \begin{vmatrix} |a_1b_2| & |b_1c_2| & |c_1d_2| \\ |a_1b_3| & |b_1c_3| & |c_1d_3| \\ |a_1b_4| & |b_1c_4| & |c_1d_4| \end{vmatrix}.$$

The complementary of this with respect to $|a_1b_2c_3d_4|$ is

$$|a_2c_3d_4||a_2b_3d_4| = \begin{vmatrix} |c_3d_4| & |a_3d_4| & |a_3b_4| \\ |c_2d_4| & |a_2d_4| & |a_2b_4| \\ |c_2d_3| & |a_2d_3| & |a_2b_3| \end{vmatrix}, \qquad \text{......(A)}$$

an identity not hitherto noticed, but which when known can be established otherwise. The complementary with respect to $|a_1b_2c_3d_4e_5|$ is

$$e_5|a_2c_3d_4e_5||a_2b_3d_4e_5| = \begin{vmatrix} |c_3d_4e_5| & |a_3d_4e_5| & |a_3b_4e_5| \\ |c_2d_4e_5| & |a_2d_4e_5| & |a_3b_4e_5| \\ |c_2d_3e_5| & |a_2d_3e_5| & |a_2b_3e_5| \end{vmatrix} \qquad \text{......(B)}$$

and by inserting "$f_6g_7\ldots$" after every e_5 in this we have a result which includes (A) and (B), viz. the complementary with respect to $|a_1b_2 \ldots g_7 \ldots|$. A still more general theorem will be got by taking the complementary of the theorem of § 93, of which that of § 53 is a particular case.

The student will find it instructive to take every theorem to which the law is applicable and find the complementary theorem. Even where no new result is obtained, some new tie of relationship may be made apparent.

EXAMPLE. Prove that

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} (ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = \begin{vmatrix} 0 & x & y & z \\ x & \mathsf{A} & \mathsf{H} & \mathsf{G} \\ y & \mathsf{H} & \mathsf{B} & \mathsf{F} \\ z & \mathsf{G} & \mathsf{F} & \mathsf{C} \end{vmatrix},$$

where A, H, ... stand for the complementary minors of a, h, ... in the first determinant.

Developing the right-hand member by § 62 as a quadratic in x, y, z, ..., we have the co-factor of x^2 in it

$$= \begin{vmatrix} \mathsf{B} & \mathsf{F} \\ \mathsf{F} & \mathsf{C} \end{vmatrix} = a \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \qquad \text{(§ 96)}$$

as it should be. Similarly it is seen that the co-factors of y^2, z^2, ... are the same in both members.

EXERCISES. SET XIII.

1. Show that the sum of the numbers indicating the rows and columns from which the elements of a minor are taken and the sum of the corresponding numbers in the case of the complementary minor are either both even or both odd.

Resolve the following into determinant factors:—

2. $\begin{vmatrix} z_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 & y_2 & x_2 \\ 0 & 0 & z_3 & y_3 & 0 & 0 \\ z_3 & z_3 & z_2 & y_2 & y_3 & y_3 \\ 0 & 0 & z_1 & y_1 & 0 & 0 \\ z_2 & z_1 & 0 & 0 & y_1 & x_1 \end{vmatrix}$. 3. $\begin{vmatrix} z_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 & y_2 & 0 \\ 0 & 0 & z_3 & y_3 & 0 & x_3 \\ 0 & z_3 & z_2 & y_2 & y_3 & x_2 \\ z_3 & 0 & z_1 & y_1 & 0 & x_1 \\ z_2 & z_1 & 0 & 0 & y_1 & 0 \end{vmatrix}$. 4. $\begin{vmatrix} a_1 & 0 & 0 & a_4 & 0 & 0 \\ 0 & a_2 & 0 & 0 & a_5 & 0 \\ 0 & 0 & a_3 & 0 & 0 & a_6 \\ b_1 & 0 & 0 & b_4 & 0 & 0 \\ 0 & b_2 & 0 & 0 & b_5 & 0 \\ 0 & 0 & b_3 & 0 & 0 & b_6 \end{vmatrix}$.

5. Use § 77 to show that

$$\begin{vmatrix} 0 & a_2 & 0 & a_4 & 0 \\ b_1 & 0 & b_3 & 0 & b_5 \\ 0 & c_2 & 0 & c_4 & 0 \\ d_1 & 0 & d_3 & 0 & d_5 \\ 0 & e_2 & 0 & e_4 & 0 \end{vmatrix} = 0.$$

6. Show that if m elements of one row of a determinant of the n^{th} order contain a common factor, which is also contained in the corresponding elements of other $n-m$ rows, this factor is a factor of the determinant.

7. Use § 77 to show that

$$\begin{vmatrix} a_2 & b_2 & c_2 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 \\ a_1A_1 & b_1A_1 & c_1A_1 & A_2 & A_3 \\ a_1B_1 & b_1B_1 & c_1B_1 & B_2 & B_3 \\ a_1C_1 & b_1C_1 & c_1C_1 & C_2 & C_3 \end{vmatrix} = |A_1B_2C_3|\,|a_1b_2c_3|.$$

8. Expand in a series of terms of the form $(p_1-p_2)(a_1-a_2)x_1x_2$ the determinant

$$\begin{vmatrix} x_4 & 0 & 0 & 0 & 1 & a_1 \\ 0 & x_3 & 0 & 0 & 1 & a_2 \\ 0 & 0 & x_2 & 0 & 1 & a_3 \\ 0 & 0 & 0 & x_1 & 1 & a_4 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ p_1 & p_2 & p_3 & p_4 & 0 & 0 \end{vmatrix}.$$

9. Show that

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ b_4 & b_3 & b_2 & b_1 \\ a_4 & a_3 & a_2 & a_1 \end{vmatrix} = \begin{vmatrix} a_1+a_4 & a_2+a_3 \\ b_1+b_4 & b_2+b_3 \end{vmatrix} \begin{vmatrix} a_1-a_4 & a_2-a_3 \\ b_1-b_4 & b_2-b_3 \end{vmatrix}.$$

10. If a determinant of the n^{th} order be developed as a sum of products, the factors of which are a minor of the α^{th} order, a minor of the β^{th} order, and a minor of the γ^{th} order, how many terms will there be in the development?

11. Deduce the identity at the end of § 87 from three of those on p. 124.

12. Prove that

$$|a_3 b_4| \begin{vmatrix} 0 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - |a_2 b_3| \begin{vmatrix} 0 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} = a_3 b_1 |a_2 b_3 c_4|.$$

13. Prove that the product of the adjugates of two determinants of the same order is equal to the adjugate of the product of the said determinants.

14. Prove the identities (2) p. 119 and (3) p. 120 by altering the forms of the factors and using the ordinary multiplication theorem (§ 67).

15. Prove that if $|a_{1n}| = 0$, then

$$A_{r1}A_{1r} : A_{r2}A_{2r} : A_{r3}A_{3r} : \ldots\ldots :: A_{11} : A_{22} : A_{33} : \ldots\ldots$$

16. Prove that

$$a_1|a_2 b_3 c_4| - a_2|a_1 b_3 c_4| = |a_1 c_2||a_3 b_4| - |a_1 b_2||a_3 c_4|.$$

17. If $|a_0 d_1| + |b_0 c_1| = |a_0 d_2| + |b_0 c_2| = |a_1 d_3| + |b_1 c_3| = |a_2 d_3| + |b_2 c_3| = 0$, then

$$|a_0 d_3||a_1 d_2| = |b_0 c_3||b_1 c_2|.$$

18. Prove that if $|a_{1n}| = 1$, then $|a_{1n}|$ and $|A_{1n}|$ are mutually adjugate.

19. Prove the theorem of § 72 after the manner of §§ 82, 83.

20. Prove that

$$|a_2 b_3| \begin{vmatrix} 0 & 0 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} - a_2|a_3 b_4||b_1 c_2 d_3| = a_3 \begin{vmatrix} a_2 & a_3 & 0 & 0 & a_4 \\ b_2 & b_3 & 0 & 0 & b_4 \\ b_2 & b_3 & b_1 & b_2 & b_4 \\ c_2 & c_3 & c_1 & c_2 & c_4 \\ d_2 & d_3 & d_1 & d_2 & d_4 \end{vmatrix}.$$

21. Using the notation of § 98, prove that

$$|A_{1n}| = \frac{\mathfrak{m}_r \mathfrak{m}_{n-r+1}}{M'_{n-r} M'_{r-1}}.$$

22. Prove that if in $|a_1 b_2 c_3 d_4|$ the co-factor of b_3 be equal to zero, then $|a_1 b_2 c_3 d_4||b_1 d_4| = |a_1 c_3 d_4||a_1 b_2 d_4|$. State the same theorem in regard to $|a_{1n}|$.

23. If $|a_0 d_1| + |b_0 c_1| = 0 = |a_0 d_3| + |b_0 c_2|$, then $|a_0 b_1 c_2 d_3| = \left\{ |a_0 d_3| + |b_0 c_3| \right\} \left\{ |a_1 d_2| + |b_1 c_2| \right\}$.

24. Prove that

$$|a_1 c_2 e_3||b_1 d_2 e_3| = |a_1 d_2 e_3||b_1 c_2 e_3| + |a_1 b_2 e_3||c_1 d_2 e_3|.$$

25. Prove that if $|a_1 b_2 c_3 d_4| = 0$, then

$$|a_1 b_3||a_1 c_2 d_4| = |a_1 b_2||a_1 c_3 d_4| + |a_1 b_4||a_1 c_2 d_3|.$$

26. Resolve into determinant-factors of the second order the determinant

$$\begin{vmatrix} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{vmatrix}.$$

27. From a determinant D_1 the minors consisting of four adjacent elements are taken in order to be the elements of a new determinant D_2: in D_2 every minor consisting of four adjacent elements is divided by the corresponding element in the minor of D_1 obtained by deleting the first and last rows and columns, and the quotients are placed in order to form D_3: in like manner D_4 is obtained from the elements of D_3 and D_2: and so on. Prove that the final result is equal to D_1.

28. Use the process of Ex. 27 to perform Exs. 1, 2, 3, 4 of Set X.

§ 100. *If* $\iota_1\iota_2\iota_3\ldots\iota_n = 1$ *and* $\iota_1, \iota_2, \ldots$ *be symbols subject to the laws of ordinary algebra except that* $\iota_r\iota_s = \iota_s\iota_r$ *and* $\iota_r^2 = 0$, *then*

$$|a_{1n}| = (a_{11}\iota_1 + a_{12}\iota_2 + \ldots + a_{1n}\iota_n)(a_{21}\iota_1 + a_{22}\iota_2 + \ldots + a_{2n}\iota_n)\ldots\ldots$$
$$\ldots(a_{n1}\iota_1 + a_{n2}\iota_2 + \ldots + a_{nn}\iota_n).$$

Writing in a column the factors

$$\begin{aligned} &a_{11}\iota_1 + a_{12}\iota_2 + a_{13}\iota_3 + \ldots + a_{1n}\iota_n, \\ &a_{21}\iota_1 + a_{22}\iota_2 + a_{23}\iota_3 + \ldots + a_{2n}\iota_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a_{n1}\iota_1 + a_{n2}\iota_2 + a_{n3}\iota_3 + \ldots + a_{nn}\iota_n, \end{aligned}$$

it becomes evident that the identity to be established is but a symbolical statement of the definition (§ 16) of a determinant. For, firstly, owing to the constitution of the factors, the product must consist of all terms of the form

$$a_{1r}a_{2s}a_{3u}\ldots a_{nz} \times \iota_r\iota_s\iota_u\ldots\iota_z,$$

which can be got by taking one and only one element from each row of the determinant: secondly, the condition $\iota_r^2 = 0$ necessitates the disappearance from this of every term containing two or more elements from the same column; and, thirdly, the conditions $\iota_1\iota_2\iota_3 \ldots \iota_n = 1$, $\iota_r\iota_s = -\iota_s\iota_r$ ensure that the sign-factor of any term shall be $+1$ or -1 according as the number of inversions of order in the suffixes of its ι's, that is, in the second suffixes of its a's, is even or odd.

EXAMPLE:— $(a\iota_1 + b\iota_2 + c\iota_3)\ (d\iota_1 + e\iota_2 + f\iota_3)\ (g\iota_1 + h\iota_2 + k\iota_3)$

$$\begin{aligned}
&= (bd\iota_2\iota_1 + cd\iota_3\iota_1 + ae\iota_1\iota_2 + ce\iota_3\iota_2 + af\iota_1\iota_3 + bf\iota_2\iota_3)\ (g\iota_1 + h\iota_2 + k\iota_3), \\
&= \{(ae - bd)\iota_1\iota_2 + (af - cd)\iota_1\iota_3 + (bf - ce)\iota_2\iota_3\}\ (g\iota_1 + h\iota_2 + k\iota_3), \\
&= g(bf - ce)\iota_2\iota_3\iota_1 + h(af - cd)\iota_1\iota_3\iota_2 + k(ae - bd)\iota_1\iota_2\iota_3, \\
&= \{g(bf - ce) - h(af - cd) + k(ae - bd\}\,\iota_1\iota_2\iota_3, \\
&= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}.
\end{aligned}$$

§ 101. In coming thus to a close with the general theorems on determinants, the student will find it instructive to retrace his steps, and, taking in all the theorems at one view, to observe how simply and naturally they flow out of the definition, and how, indeed, when the definition has become thoroughly known to him, several of the theorems appear self-evident. Other definitions might have been started from with the like result; in fact, owing to the lengthiness of that here adopted, and more especially to the cumbersomeness of its 'rule of signs,' attempts to build up the theory on a different basis have more than once been made. Of course, in any two of such differently founded systems what is given as the definition in the one appears, or might appear, as a theorem in the other. The theorems of the present work which have been taken as the definition by other writers are those of §§ 46, 100, 118. A definition has also been based on the theorem involved in Ex. 3, p. 75.

§ 102. Two notations remain to be noticed, which have not been employed in the preceding, but which are often found in writings on determinants. The first of these is important, and may be viewed as the next step in advance after the notations of §§ 20, 56. It consists in omitting the a's in writing the elements of $|a_{1n}|$; so that (r,s) or (rs) is put for $a_{r,s}$ and

$$\begin{vmatrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{vmatrix}_a \quad \text{or} \quad \begin{vmatrix} (11) & (12) & (13) \\ (21) & (22) & (23) \\ (31) & (32) & (33) \end{vmatrix} \quad \text{for } |a_{13}|.$$

Sylvester, who calls this his *umbral* notation, writes also

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} \quad \text{for} \quad |a_{13}|,$$

and generally

$$\begin{vmatrix} a & b & c \ldots\ldots \\ \alpha & \beta & \gamma \ldots\ldots \end{vmatrix} \quad \text{for} \quad |x_{a\alpha} x_{b\beta} x_{c\gamma} \ldots\ldots|.$$

§ 103. From the *rectangular array* of elements or *matrix*,

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{matrix}$$

ten determinants whose columns are columns of the array can be formed. To indicate, if need be, that these all vanish, it is customary to write

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{Vmatrix} = 0.$$

To the left-hand member by itself no meaning is attached.

CHAPTER III.

DETERMINANTS OF SPECIAL FORM.

§ 104. Determinants, which are of special form by reason of a number of the elements being interdependent, *e.g.* the determinant

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix},$$

or by reason of a number of the elements having particular values, *e.g.* the determinant

$$\begin{vmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{vmatrix},$$

have properties peculiar to themselves. The more important of such special forms, the simplest first, will now be referred to, several of them under the specific designations which it has been found convenient to give them.

CONTINUANTS.

§ 105. A determinant which has the elements lying outside the principal diagonal and the two bordering minor diagonals each equal to zero, and which has the elements of one of these minor diagonals each equal to negative unity, is called a CONTINUANT. Thus

$$\begin{vmatrix} a_1 & b_1 & 0 & 0 \\ -1 & a_2 & b_2 & 0 \\ 0 & -1 & a_3 & b_3 \\ 0 & 0 & -1 & a_4 \end{vmatrix}$$

is a continuant of the fourth order.

The origin of the name will appear from an identity to be established later (pp. 157, 158).

§ 106. The minor diagonal whose elements are not fixed by the definition (e.g. b_1, b_2, b_3 in the above) may be spoken of as the *variable minor diagonal*, and the other (–1, –1, –1) as the *invariable minor diagonal;* and since the orderly change of rows into columns only transposes these two diagonals, it is immaterial (§ 24) on which side of the principal diagonal either of them is written.

A continuant being evidently a function of the elements of the principal diagonal and the variable minor diagonal, and of these alone, a shorter notation may be adopted for it which shall take note only of these elements: The above continuant, for example, may be written

$$K\left(\begin{matrix} & b_1 & & b_2 & & b_3 & \\ a_1 & & a_2 & & a_3 & & a_4 \end{matrix}\right).$$

§ 107. *A determinant differing from a continuant only in having, instead of the invariable minor diagonal, a diagonal with non-zero elements other than* –1 *may be expressed as a continuant by changing each such element into* –1 *and altering the corresponding element in the variable minor diagonal so that the product of the two elements may remain unchanged.*

Let the determinant be

$$\begin{vmatrix} a_1 & b_1 & 0 & 0 & 0\dots \\ c_1 & a_2 & b_2 & 0 & 0\dots \\ 0 & c_2 & a_3 & b_3 & 0\dots \\ 0 & 0 & c_3 & a_4 & b_4\dots \\ 0 & 0 & 0 & c_4 & a_5\dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} \quad \text{or} \quad G,$$

the elements $c_1, c_2, c_3, \dots.$ occurring where $-1, -1, -1, \dots.$ occur in a continuant.

Multiplying the first column by $-1/c_1$ and the first row by $-c_1$, the second column by $-1/c_2$ and the second row by $-c_2$, and so on, we have the continuant

$$\begin{vmatrix} a_1 & -b_1c_1 & 0 & 0 & 0 & \dots.. \\ -1 & a_2 & -b_2c_2 & 0 & 0 & \dots.. \\ 0 & -1 & a_3 & -b_3c_3 & 0 & \dots.. \\ 0 & 0 & -1 & a_4 & -b_4c_4 & \dots.. \\ 0 & 0 & 0 & -1 & a_5 & \dots.. \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

But as each pair of these operations leaves the determinant unaltered in substance, it follows that the resulting continuant is equal to G; and as

$$(-b_1c_1) \times (-1) = b_1 \times c_1, \ (-b_2c_2) \times (-1) = b_2 \times c_2, \ \dots\dots,$$

the theorem is established.

§ 108. *Any continuant is equal to the continuant got by reversing the order of the elements in the two variable diagonals;*

$$\text{i.e.,} \quad K\begin{pmatrix} & b_1 & & b_2 & & b_{n-1} & \\ a_1 & & a_2 & \dots\dots & a_{n-1} & & a_n \end{pmatrix} = K\begin{pmatrix} & b_{n-1} & & & & b_1 & \\ a_n & & a_{n-1} & \dots\dots & a_2 & & a_1 \end{pmatrix}.$$

This follows at once from §§ 24, 39.

Example:—

$$\begin{vmatrix} a_1 & b_1 & 0 & 0 \\ -1 & a_2 & b_2 & 0 \\ 0 & -1 & a_2 & b_3 \\ 0 & 0 & -1 & a_4 \end{vmatrix} = \begin{vmatrix} a_1 & -1 & 0 & 0 \\ b_1 & a_2 & -1 & 0 \\ 0 & b_3 & a_3 & -1 \\ 0 & 0 & b_3 & a_4 \end{vmatrix} \quad (\S\ 24)$$

$$= \begin{vmatrix} a_4 & b_3 & 0 & 0 \\ -1 & a_3 & b_2 & 0 \\ 0 & -1 & a_2 & b_1 \\ 0 & 0 & -1 & a_1 \end{vmatrix}. \quad (\S\ 39)$$

§ 109. *Any continuant may be expressed in terms of continuants of lower order whose diagonals are portions of the diagonals of the original continuant; thus—*

$$K\begin{pmatrix} b_1 & & b_{n-1} \\ a_1 a_2 \ldots a_{n-1} a_n \end{pmatrix} = a_1 K\begin{pmatrix} b_2 & & b_{n-1} \\ a_2 a_3 \ldots a_{n-1} a_n \end{pmatrix} + b_1 K\begin{pmatrix} b_3 & b_{n-1} \\ a_3 \ldots a_{n-1} a_n \end{pmatrix},$$

or, more generally,

$$K\begin{pmatrix} b_1 & & b_{n-1} \\ a_1 a_2 \ldots a_{n-1} a_n \end{pmatrix} = K\begin{pmatrix} b_1 & b_{p-1} \\ a_1 a_2 \ldots a_p \end{pmatrix} K\begin{pmatrix} b_{p+1} & b_{n-1} \\ a_{p+1} & \ldots a_n \end{pmatrix}$$
$$+ b_p K\begin{pmatrix} b_1 & b_{p-2} \\ a_1 \ldots a_{p-1} \end{pmatrix} K\begin{pmatrix} b_{p+2} & b_{n-1} \\ a_{p+2} & \ldots a_n \end{pmatrix},$$

where $p < n$.

Expressing the continuant

$$\begin{vmatrix} a_1 & b_1 & 0 & 0 \ldots\ldots\ldots & 0 \\ -1 & a_2 & b_2 & 0 \ldots\ldots\ldots & 0 \\ 0 & -1 & a_3 & b_3 \ldots\ldots\ldots & 0 \\ 0 & 0 & -1 & a_4 \ldots\ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ 0 & 0 & 0 & 0 \ldots \;\; -1 & a_n \end{vmatrix}$$

in terms of the elements of the first row and their complementary minors, we have

$$a_1 \begin{vmatrix} a_2 & b_2 & 0 \ldots\ldots\ldots & 0 \\ -1 & a_3 & b_3 \ldots\ldots\ldots & 0 \\ 0 & -1 & a_4 \ldots\ldots\ldots & 0 \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots \\ 0 & 0 & 0 \ldots \;\; -1 & a_n \end{vmatrix} - b_1 \begin{vmatrix} -1 & b_1 & 0 \ldots\ldots\ldots & 0 \\ 0 & a_3 & b_3 \ldots\ldots\ldots & 0 \\ 0 & -1 & a_4 \ldots\ldots\ldots & 0 \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots \\ 0 & 0 & 0 \ldots \;\; -1 & a_n \end{vmatrix},$$

and expanding the latter determinant of this in the same way, there results

$$a_1 \begin{vmatrix} a_2 & b_2 & 0 \ldots\ldots\ldots & 0 \\ -1 & a_3 & b_3 \ldots\ldots\ldots & 0 \\ 0 & -1 & a_4 \ldots\ldots\ldots & 0 \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots \\ 0 & 0 & 0 \ldots \;\; -1 & a_n \end{vmatrix} + b_1 \begin{vmatrix} a_3 & b_3 \ldots\ldots\ldots & 0 \\ -1 & a_4 \;\; b_4 \ldots\ldots & 0 \\ \ldots & \ldots\ldots\ldots & \ldots \\ 0 & 0 \ldots \;\; -1 & a_n \end{vmatrix},$$

or

$$a_1 K\left(\begin{array}{ccccc} & b_2 & & b_{n-1} & \\ a_2 & & a_3 \ldots\ldots & & a_n \end{array}\right) + b_1 K\left(\begin{array}{ccccc} & b_3 & & b_{n-1} & \\ a_3 & & a_4 \ldots\ldots & & a_n \end{array}\right),$$

as was to be shown.

Again, writing the given continuant in fuller form, thus—

$$\begin{vmatrix} a_1 & b_1 & 0 & 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 0 & 0 \\ -1 & a_2 & b_2 & 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 0 & 0 \\ 0 & -1 & a_3 & b_3 \;\; 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots \\ 0 & 0 \ldots\ldots\ldots & -1 & a_p \quad b_p \quad 0 \ldots\ldots\ldots\ldots\ldots\ldots & 0 & 0 \\ 0 & 0 \ldots\ldots\ldots\ldots & & -1 \quad a_{p+1} \quad b_{p+1} \quad 0 \ldots\ldots\ldots & 0 & 0 \\ 0 & 0 \ldots\ldots\ldots\ldots\ldots & & \quad -1 \quad a_{p+2} \quad b_{p+2} \; 0 \ldots\ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots \\ 0 & 0 \ldots\ldots\ldots\ldots\ldots & & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & -1 & a_n \end{vmatrix},$$

let us try to find all the minors of the p^{th} degree which can be formed from the first p rows, so that, multiplying every such minor by its complementary, we may express the continuant in terms of the products thus obtained.

First it will be seen that we need only consider the minors of this kind which have for their first $p-1$ columns

$$\begin{array}{cccccc}
a_1 & b_1 & 0 & 0 & \dots\dots\dots & \\
-1 & a_2 & b_2 & 0 & \dots\dots\dots & \\
0 & -1 & a_3 & b_3 & \dots\dots\dots & \\
\dots & \dots & \dots & \dots & \dots\dots\dots & \\
0 & 0 & 0 & 0 & \dots & a_{p-1} \\
0 & 0 & 0 & 0 & \dots & -1 ,
\end{array}$$

for those which do not contain any one of these columns must have complementaries which vanish. Secondly, there are only two of the former sort which do not themselves vanish, viz. those whose last columns are—

$$\begin{array}{c} 0 \\ 0 \\ \vdots \\ b_{p-1} \\ a_p \end{array} \qquad \text{and} \qquad \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ b_p . \end{array}$$

the last columns of all the others containing nothing but zeros. Hence we have the following expression for the continuant:—

$$\begin{vmatrix}
a_1 & b_1 & 0 & \dots\dots\dots\dots \\
-1 & a_2 & b_2 & \dots\dots\dots\dots \\
0 & -1 & a_3 & b_3 \dots\dots \\
\dots & \dots & \dots & \dots\dots\dots\dots \\
0 & 0 \dots\dots\dots & -1 & a_p
\end{vmatrix} \times \begin{vmatrix}
a_{p+1} & b_{p+1} & 0 & \dots\dots\dots\dots \\
-1 & a_{p+2} & b_{p+2} & \dots\dots\dots \\
0 & -1 & a_{p+3} & \dots\dots\dots \\
\dots & \dots & \dots & \dots\dots\dots\dots \\
0 & 0 \dots\dots\dots & -1, & a
\end{vmatrix}$$

$$
-\begin{vmatrix}
a_1 & b_1 & 0 & \cdots\cdots & & \\
-1 & a_2 & b_2 & \cdots\cdots & & \\
0 & -1 & a_3 & b_3 \cdots\cdots & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots\cdots & & a_{p-1} & 0 \\
0 & 0 & \cdots\cdots & & -1 & b_p
\end{vmatrix}
\times
\begin{vmatrix}
-1 & b_{p+1} & 0 & \cdots\cdots & \\
0 & a_{p+2} & b_{p+2} & \cdots\cdots & \\
0 & -1 & a_{p+3} & \cdots\cdots & \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots\cdots & -1 & a_n
\end{vmatrix}.
$$

But each of the last two determinants is expressible as the product of one element and its complementary, and these changes being made, we have the result required.

EXAMPLES:—

$$
\begin{aligned}
K\left({}_a{}^{\alpha}{}_b{}^{\beta}{}_c{}^{\gamma}{}_d{}^{\delta}{}_e\right) &= aK\left({}_b{}^{\beta}{}_c{}^{\gamma}{}_d{}^{\delta}{}_e\right) + \alpha K\left({}_c{}^{\gamma}{}_d{}^{\delta}{}_e\right) \\
&= K\left({}_a{}^{\alpha}{}_b\right) K\left({}_c{}^{\gamma}{}_d{}^{\delta}{}_e\right) + \beta K(a)\, K\left({}_d{}^{\delta}{}_e\right).
\end{aligned}
$$

§ 110. The first of the results in the preceding is important as affording an easy means of finding for a given continuant its ordinary expansion in non-determinant form. Thus, if the continuant be

$$K(\mathrm{A}^a\mathrm{B}^b\mathrm{C}^c\mathrm{D}^d\mathrm{E}),$$

we first note that

$$
\begin{aligned}
K(\mathrm{D}^d\mathrm{E}) &= \mathrm{DE} + d; \\
K(\mathrm{C}^c\mathrm{D}^d\mathrm{E}) &= \mathrm{C}K(\mathrm{D}^d\mathrm{E}) \quad + cK(\mathrm{E}) \\
&= \mathrm{C}(\mathrm{DE} + d) + c\mathrm{E} \\
&= \mathrm{CDE} + d\mathrm{C} + c\mathrm{E};
\end{aligned}
$$

thirdly, that

$$
\begin{aligned}
K(\mathrm{B}^b\mathrm{C}^c\mathrm{D}^d\mathrm{E}) &= \mathrm{B}K(\mathrm{C}^c\mathrm{D}^d\mathrm{E}) + bK(\mathrm{D}^d\mathrm{E}) \\
&= \mathrm{B}(\mathrm{CDE} + d\mathrm{C} + c\mathrm{E}) + b(\mathrm{DE} + d) \\
&= \mathrm{BCDE} + \mathrm{B}d\mathrm{C} + \mathrm{B}c\mathrm{E} + b\mathrm{DE} + bd;
\end{aligned}
$$

and fourthly, that

$$\begin{aligned} K(\mathrm{A}^a\mathrm{B}^b\mathrm{C}^c\mathrm{D}^d\mathrm{E}) &= \mathrm{A}K(\mathrm{B}^b\mathrm{C}^c\mathrm{D}^d\mathrm{E}) + aK(\mathrm{C}^c\mathrm{D}^d\mathrm{E}) \\ &= \mathrm{A}(\mathrm{BCDE} + \mathrm{BC}d + \mathrm{BE}c + \mathrm{DE}b + bd) \\ &\quad + a(\mathrm{CDE} + \mathrm{C}d + \mathrm{E}c) \\ &= \mathrm{ABCDE} + \mathrm{ABC}d + \mathrm{ABE}c + \mathrm{ADE}b + \mathrm{CDE}a \\ &\qquad + \mathrm{A}bd + \mathrm{C}ad + \mathrm{E}ac, \end{aligned}$$

each of the successive results being got by the help of the preceding results.

§ 111. *The product of a continuant and one of its continuant minors is expressible by means of four other minors of the same kind; thus—*

$$\begin{aligned} &K\binom{b_1 \;\dots\; b_{n-1}}{a_1\, a_2 \dots\; a_n} K\binom{b_h \dots b_{p-1}}{a_h \;\dots\; a_p} \\ &\qquad = K\binom{b_1 \dots b_{p-1}}{a_1 \;\dots\; a_p} K\binom{b_h \dots b_{n-1}}{a_h \;\dots\; a_n} \\ &\qquad\quad + (-1)^{p-h+1} b_{h-1} b_h \dots b_p K\binom{b_1 \dots b_{h-3}}{a_1 \dots a_{h-2}} K\binom{b_{p+2} \dots b_{n-1}}{a_{p+2} \;\dots\; a_n}, \end{aligned}$$

where, of course, $h < p < n$.

Writing out the continuant $K\binom{b_1 \;\dots\; b_{n-1}}{a_1 a_2 \dots\; a_n}$ in full as a determinant, and, in accordance with § 90, forming from its first p rows all the minors of the p^{th} order which contain the minor $K\binom{b_h \dots b_{p-1}}{a_h \dots\; a_p}$, and taking with each of these its proper co-factor, it is seen that, on account of zero columns, only two of the products do not vanish. One of the two is $K\binom{b_1 \dots b_{p-1}}{a_1 \;\dots\; a_p} K\binom{b_h \dots b_{n-1}}{a_h \;\dots\; a_n}$, the sign being $+$: the factors of the other are transformable into $b_{h-1} b_h \dots b_p K\binom{b_1 \dots b_{h-3}}{a_1 \dots a_{h-2}}$ and $-K\binom{b_{h+2} \dots b_{n-1}}{a_{p+2} \dots\; a_n}$ by § 46, the sign proper to the product being $(-1)^{p-h+2}$. Thus the theorem is established.

An important particular case is that for which $p = n - 1$ and $h = 2$, the theorem then being

$$K\begin{pmatrix} b_1 \ldots\ldots b_{n-1} \\ a_1 a_2 \ldots\ldots a_n \end{pmatrix} K\begin{pmatrix} b_2 \ldots b_{n-2} \\ a_2 \ldots a_{n-1} \end{pmatrix} = K\begin{pmatrix} b_1 \ldots b_{n-2} \\ a_1 \ldots a_{n-1} \end{pmatrix} K\begin{pmatrix} b_2 \ldots b_{n-1} \\ a_2 \ldots\ldots a_n \end{pmatrix}$$
$$+ (-1)^n b_1 b_2 \ldots b_{n-1}.$$

Example:—

$$K(\mathrm{B}^b \mathrm{C}^c \mathrm{D}^d \mathrm{E})\, K(\mathrm{C}^c \mathrm{D}) = K(\mathrm{B}^b \mathrm{C}^c \mathrm{D})\, K(\mathrm{C}^c \mathrm{D}^d \mathrm{E}) + bcd,$$

the truth of which may be tested by what is given in § 110.

§ 112. When each element of the variable minor diagonal is unity, the continuant may be called *simple*: and in denoting such continuants by means of the symbolism explained in § 106, we may agree to omit the elements of this diagonal, so that, for example,

$$K(a\ b\ c\ d) = K(a^1 b^1 c^1 d^1).$$

§ 113. *Two simple continuants, the elements of whose principal diagonals are the positive integers* $\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_{n-1}$ *and* $\mathrm{a}_1, \mathrm{a}_2, \ldots, \mathrm{a}_n$ *respectively, are prime to each other.*

$$K(a_1, a_2, \ldots, a_n) = a_n K(a_1, a_2, \ldots, a_{n-1}) + K(a_1, a_2, \ldots, a_{n-2});$$

$$\begin{aligned}
\therefore \text{ G.C.M. of } & K(a_1, a_2, \ldots, a_n) \text{ and } K(a_1, a_2, \ldots, a_{n-1}) \\
&= \text{G.C.M. of } K(a_1, a_2, \ldots, a_{n-1}) \text{ and } K(a_1, a_2, \ldots, a_{n-2}) \\
&= \text{G.C.M. of } K(a_1, a_2, \ldots, a_{n-2}) \text{ and } K(a_1, a_2, \ldots, a_{n-3}) \\
&= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= \text{G.C.M. of } K(a_1 a_2) \text{ and } K(a_1) \\
&= 1.
\end{aligned}$$

Example 1. Prove the identity

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cdots + \cfrac{b_{n-1}}{a_n}}} = \frac{K\begin{pmatrix} b_1 & b_2 & & b_{n-1} & \\ a_1 & a_2 & a_3 \ldots\ldots & & a_n \end{pmatrix}}{K\begin{pmatrix} b_2 & & b_{n-1} & \\ a_2 & a_3 \ldots\ldots & & a_n \end{pmatrix}}.$$

$$\text{The right-hand member} = \frac{a_1 K\begin{pmatrix} b_2 \,.\,.\, b_{n-1} \\ a_2 \ a_3 \,.\,.\,.\,.\, a_n \end{pmatrix} + b_1 K\begin{pmatrix} b_3 \,.\,.\, b_{n-1} \\ a_3 \,.\,.\,.\,.\, a_n \end{pmatrix}}{K\begin{pmatrix} b_2 \,.\,.\, b_{n-1} \\ a_2 \ a_3 \,.\,.\,.\,.\, a_n \end{pmatrix}}, \qquad (\S\, 109)$$

$$= a_1 + \frac{b_1 K\begin{pmatrix} b_3 \,.\,.\, b_{n-1} \\ a_3 \,.\,.\,.\,.\, a_n \end{pmatrix}}{K\begin{pmatrix} b_2 \,.\,.\, b_{n-1} \\ a_2 \ a_3 \,.\,.\,.\,.\, a_n \end{pmatrix}},$$

$$= a_1 + \cfrac{b_1}{\cfrac{K\begin{pmatrix} b_2 \,.\,.\, b_{n-1} \\ a_2 \ a_3 \,.\,.\,.\,.\, a_n \end{pmatrix}}{K\begin{pmatrix} b_3 \,.\,.\, b_{n-1} \\ a_3 \,.\,.\,.\,.\, a_n \end{pmatrix}}}.$$

Next, treating the quotient of the two continuants here as the original quotient was treated, we find

$$\text{the right-hand member} = a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{Q_3}},$$

$$\text{where } Q_3 = \frac{K\begin{pmatrix} b_3 \,.\,.\, b_{n-1} \\ a_3 \,.\,.\,.\,.\, a_n \end{pmatrix}}{K\begin{pmatrix} b_4 \,.\,.\, b_{n-1} \\ a_4 \,.\,.\,.\,.\, a_n \end{pmatrix}}.$$

Continuing this process, we at length come to the quotient

$$\frac{K\begin{pmatrix} b_{n-1} \\ a_{n-1} \quad a_n \end{pmatrix}}{K(a_n)},$$

$$\text{which} = \frac{a_{n-1}\, a_n + b_{n-1}}{a_n},$$

$$= a_{n-1} + \frac{b_{n-1}}{a_n}.$$

Hence the identity is established.

EXAMPLE 2. Prove that the periodic continued fraction

$$A + \frac{b_1}{\underset{*}{a_1}} + \frac{b_2}{a_2} + \frac{b_3}{a_3} + \dots\dots + \frac{b_3}{a_2} + \frac{b_2}{a_1} + \frac{b_1}{\underset{*}{2A}} + \dots\dots = \sqrt{\frac{K\begin{pmatrix} b_1 \ b_2 \,.\,.\,.\,.\, b_2 \ b_1 \\ A \ \ a_1 \ a_2 \,.\,.\,.\,.\, a_2 \ a_1 \ \ A \end{pmatrix}}{K\begin{pmatrix} b_2 \,.\,.\,.\,.\, b_2 \\ a_1 \ a_2 \,.\,.\,.\,.\, a_2 \ a_1 \end{pmatrix}}}$$

where the asterisks are used like the superposed dots in the notation of decimal fractions to indicate the recurring portion or period.

Denoting the left-hand number by x, the portion which follows the second asterisk will be $x - A$, hence we have

$$x = A + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \dots\dots + \frac{b_3}{a_2} + \frac{b_2}{a_1} + \frac{b_1}{2A + x - A}$$

$$\therefore \quad x = \frac{K\begin{pmatrix} & b_1 & & b_2 & & \cdots & & b_2 & & b_1 & \\ A & & a_1 & & a_2 & \cdots & a_2 & & a_1 & & A+x \end{pmatrix}}{K\begin{pmatrix} & b_2 & & & & b_2 & & b_1 & \\ a_1 & & a_2 & \cdots & a_2 & & a_1 & & A+x \end{pmatrix}} \qquad \text{(Example 1)}$$

$$= \frac{K\begin{pmatrix} & b_1 & & b_2 & & \cdots & & b_2 & & b_1 & \\ A & & a_1 & & a_2 & \cdots & a_2 & & a_1 & & A \end{pmatrix} + xK\begin{pmatrix} & b_1 & & b_2 & & \cdots & & b_2 & \\ A & & a_1 & & a_2 & \cdots & a_2 & & a_1 \end{pmatrix}}{K\begin{pmatrix} & b_2 & & \cdots & & b_2 & & b_1 & \\ a_1 & & a_2 & \cdots & a_2 & & a_1 & & A \end{pmatrix} + xK\begin{pmatrix} & b_2 & & \cdots & & b_2 & \\ a_1 & & a_2 & \cdots & a_2 & & a_1 \end{pmatrix}} \qquad (\S\ 109)$$

Multiplying both sides by the denominator, and rejecting a common term, there results

$$x^2 K\begin{pmatrix} & b_2 & & \cdots & & b_2 & \\ a_1 & & a_2 & \cdots & a_2 & & a_1 \end{pmatrix} = K\begin{pmatrix} & b_1 & & b_2 & & \cdots & & b_2 & & b_1 & \\ A & & a_1 & & a_2 & \cdots & a_2 & & a_1 & & A \end{pmatrix},$$

whence the identity required.

EXERCISES. SET XIV.

1. Write out the theorems regarding simple continuants which are included in those established in §§ 109, 111.

Prove the following theorems:—

2. $K\begin{pmatrix} & mb_1 & & b_2 & & \cdots\cdots \\ ma_1 & & a_2 & & a_3 & \cdots\cdots \end{pmatrix} = mK\begin{pmatrix} & b_1 & & b_2 & & \cdots\cdots \\ a_1 & & a_2 & & a_3 & \cdots\cdots \end{pmatrix}$.

3. $K(0, a_2, a_3, \dots, a_n) = K(a_3, \dots, a_n)$.

4. $K(\dots, a, b, c, 0, e, f, g, \dots) = K(\dots, a, b, c+e, f, g, \dots)$.

5. $K(\dots, a, b, c, 0, 0, 0, e, f, \dots) = K(\dots, a, b, c+e, f, \dots)$.

6. $K(0, 0, a_3, a_4, \dots, a_n) = K(a_3, a_4, \dots, a_n)$.

7. $K(\dots a, b, 0, 0, e, f, \dots) = K(\dots, a, b, e, f, \dots)$.

8. $K(a_1, a_2, \dots, a_n, a_n, \dots a_2, a_1) = K(a_1, a_2, \dots a_{n-1})^2 + K(a_1, a_2, \dots, a_n)^2$.

9. $K(a_1, a_2, \dots, a_{n-1}, a_n, a_{n-1}, \dots, a_2, a_1) = K(a_1, a_2, \dots, a_{n-1}) \times$
$\{K(a_1, a_2, \dots, a_{n-2}) + K(a_1, a_2, \dots, a_n)\}$.

10. $K\begin{pmatrix} & b_1 & & b_2 & & b_3 & & \cdots \\ 1 & & a_1 & & a_2 & & a_3, & \cdots \end{pmatrix} = K\begin{pmatrix} & b_2 & & b_3 & & \cdots \\ a_1+b_1 & & a_2 & & a_3, & \cdots \end{pmatrix}$.

11. $K\begin{pmatrix} & b_1 & & b_2 \cdots & & b_{n-1} & \\ -a_1, & & -a_2, & & \cdots -a_{n-1}, & & -a_n \end{pmatrix} = (-1)_n K\begin{pmatrix} & b_1 & & \cdots & & b_{n-1} & \\ a_1 & & a_2 & \cdots & a_{n-1} & & a_n \end{pmatrix}$.

12. $a_n K(a_1, a_2, \dots a_{n-1}, a_n, a_{n-1}, \dots a_2, a_1) = K(a_1, a_2, \dots a_n)^2 - K(a_1, a_2, \dots a_{n-2})^2$.

13. $K(a_1, a_2, \dots, a_{n-1}, 2a_n, a_{n-1}, \dots a_2, a_1) = 2K(a_1, a_2, \dots a_{n-1})\, K(a_1, a_2, \dots a_n)$.

14. $K\begin{pmatrix} & b_1 & & (b_1+a_1)b_2 & & (b_2+a_2)b_3 & & \cdots \\ 1 & & a_1 & & a_2 & & a_3 & \cdots \end{pmatrix} = (b_1+a_1)(b_2+a_2)(b_3+a_3) \dots$.

15. $K\begin{pmatrix} & b_1 & & b_2 & & b_3 \cdots b_{n-1} & \\ a_1 & & a_2 & & a_3 & \cdots\cdots & a_n \end{pmatrix} = K\left(a_1, a_2\frac{1}{b_1}, a_3\frac{b_1}{b_2}, a_4\frac{b_2}{b_1 b_3}, a_5\frac{b_1 b_3}{b_2 b_4}, \dots, a_n\frac{\dots b_{n-4} b_{n-2}}{\dots b_{n-3} b_{n-1}}\right)$
$\times \dots b_{n-3} b_{n-1}$.

16. $K(a_1x^{-1}, a_2x, a_3x^{-1}, a_4x, \dots, a_nx^{(-1)^n}) = K(a_1, a_2, a_3, \dots, a_n)$

or $= K(a_1, a_2, a_3, \dots, a_n) \times x^{-1}$

according as n is even or odd.

17. $K(a_1, a_2, a_3, \dots, a_{n-1}, a_n)$ is prime to $K(a_2, \dots, a_{n-1}, a_n)$, to $K(a_1-1, a_2, \dots, a_n)$, and to $K(a_1, a_2, \dots, a_n-1)$.

18.

$$\frac{K\begin{pmatrix} & b_1 & & b_2 & \dots & & b_2 & & b_1 & \\ A, & & a_1, & & a_2 \dots a_2, & & & a_1, & & A \end{pmatrix}}{K\begin{pmatrix} & b_2 & \dots & b_2 & \\ a_1 & & a_2 \dots a_2 & & a_1 \end{pmatrix}} = \frac{K\begin{pmatrix} & b_1 & & b_2 & \dots & b_2 & & b_1 & & b_1 & & b_2 & \dots & b_2 & & b_1 & \\ A, & & a_1, & & a_2, \dots a_2, & & a_1, & & 2A, & & a_1, & & a_2, \dots a_2, & & a_1, & & A \end{pmatrix}}{K\begin{pmatrix} & b_2 & \dots & b_2 & & b_1 & & b_1 & & b_2 & \dots & b_2 & \\ a_1, & & a_2 \dots a_2, & & a_1, & & 2A, & & a_1, & & a_2, \dots a_2, & & a_1 \end{pmatrix}}.$$

19. $$K\begin{pmatrix} & \omega + a_1x & & \omega + a_2x & & \\ a_1 + x & & a_2 & & , \dots, & a_n \end{pmatrix} = K\begin{pmatrix} & \omega + a_2x & & \omega + a_2x & & \\ a_1 & & a_2 & & , \dots & a_n + x \end{pmatrix}.$$

20. $$K\begin{pmatrix} & b_1 & & b_2 & & b_{n-1} & \\ \beta + \frac{b_1}{r} + r, & & \beta + \frac{b_2}{r}, & & \beta + \frac{b_3}{r}, \dots, & & \beta + \frac{b_n}{r} \end{pmatrix}$$

$$= K\begin{pmatrix} & b_2 & & b_3 & & b_n & \\ \beta + \frac{b_1}{r} & & \beta + \frac{b_2}{r} & & \beta + \frac{b_3}{r}, \dots, & & \beta + \frac{b_n}{r} + r \end{pmatrix}.$$

21. $$K\begin{pmatrix} & -n.1 & & -(n-1)2 & & -(n-2)3 \dots -1.n & \\ a & & a & & a & \dots\dots & a \end{pmatrix}$$

$$= (a+n)\ (a+n-2) \dots\dots (a-n+2)\ (a-n).$$

22. Prove that

$$\frac{d_1 d_2 d_3 \dots}{e_1 e_2 e_3 \dots} = 1 + \cfrac{d_1 - e_1}{e_1 - \cfrac{(d_2 - e_2) d_1 e_1}{d_1 d_2 - e_1 e_2 - \cfrac{(d_1 - e_1)(d_3 - e_3) d_2 e_2}{d_2 d_3 - e_2 e_3 - \cfrac{(d_2 - e_2)(d_4 - e_4) d_3 e_3}{d_3 d_4 - e_3 e_4 - \ddots}}}}$$

$$= \cfrac{1}{1 + \cfrac{e_1 - d_1}{d_1 - \cfrac{(e_2 - d_2) d_1 e_1}{e_1 e_2 - d_1 d_2 - \cfrac{(e_1 - d_1)(e_3 - d_3) d_2 e_2}{e_2 e_3 - d_2 d_3 - \cfrac{(e_2 - d_2)(e_4 - d_4) d_3 e_3}{e_3 e_4 - d_3 d_4 - \ddots}}}}}$$

23. Show that

$$K\begin{pmatrix} & a_1a_3 & & a_2a_4 & \\ a_1 - a_2 & & a_2 - a_3 & & a_4 - a_5 \dots \end{pmatrix} = a_2a_3 \dots (a_1 - a_2 + a_3 - a_4 + \dots);$$

and thence, with the help of Ex. 14, prove that

$$a_1 - a_2 + a_3 - \dots + (-1)^{n-1} a_n = \cfrac{a_1}{1 + \cfrac{a_2}{a_1 - a_2 + \cfrac{a_1 a_3}{a_2 - a_3 + \ddots + \cfrac{a_{n-2} a_n}{a_{n-1} - a_n}}}}.$$

24. Prove that the number of terms in the expansion of a continuant of the n^{th} order

$$= 1 + (n-1) + \frac{(n-2)(n-3)}{1.2} + \frac{(n-3)(n-4)(n-5)}{1.2.3} + \dots,$$

$$= \frac{1}{\sqrt{5}}\left\{ (\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5})^{n+1} - (\tfrac{1}{2}-\tfrac{1}{2}\sqrt{5})^{n+1} \right\}.$$

Establish the identities—

25. $$A + \frac{1}{\underset{*}{a}+}\,\frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{\underset{*}{2A}+}\dots = A + \frac{bc+2}{\underset{*}{abc+2a+c}+}\,\frac{1}{b+}\,\frac{1}{abc+2a+c+}\,\frac{bc+2}{\underset{*}{2A}}+\dots$$

26. $$A + \frac{1}{\underset{*}{a}+}\,\frac{1}{b+}\dots+\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{\underset{*}{2A}+}\dots = A + \frac{K(bc\dots cb)}{\underset{*}{K(ab\dots cb)}+}\,\frac{(-1)^{l-1}}{K(ab\dots cb)+}\,\frac{K(bc\dots cb)}{\underset{*}{2A}}+\dots$$

27. Prove that the general expression for every integer whose square root when expressed as a continued fraction with unit-numerators has $a_1, a_2, \dots a_2, a_1$ for the symmetric portion of its cycle of partial denominators is

$$\left\{\tfrac{1}{2}K(a_1\dots a_1)\mathrm{M} - (-1)^l\tfrac{1}{2}K(a_1\dots a_2)\,K(a_2\dots a_2)\right\}^2 + K(a_1\dots a_2)\mathrm{M} - (-1)^l K(a_2\dots a_2)^2,$$

l being the number of elements in the cycle.

28. Prove that the condition that any periodic continued fraction

$$A + \frac{a_1}{\underset{*}{b_1}+}\,\frac{a_2}{b_2+}\dots+\frac{a_{n-1}}{b_{n-1}+}\,\frac{a_n}{\underset{*}{b_n}+}\dots$$

may represent a quadratic surd is

$$K\left(\begin{matrix} & a_1 & & \dots & & a_{n-1} & \\ A_1 & & b_1 & \dots & b_{n-2} & & b_{n-1}\end{matrix}\right) = K\left(\begin{matrix} & a_1 & & \dots & & a_n & \\ b_1 & & b_2 & \dots & b_{n-1} & & b_n - A\end{matrix}\right).$$

ALTERNANTS.

§ 114. When the elements of the first row of a determinant are all functions of one variable, the elements of the second row the like functions of another variable, and so on, the determinant is called an *alternant:* for example,

$$\begin{vmatrix} \sin x & \cos x & 1 \\ \sin y & \cos y & 1 \\ \sin z & \cos z & 1 \end{vmatrix},$$

or the first determinant of § 104.

§ 115. Every alternant of the n^{th} order is evidently a function of n variables. To interchange two of these would be the same as to interchange two of the rows of the determinant, and therefore would have the effect of merely changing the sign of the function; and as a function having this property is known as an *alternating* function, the origin of the name alternant is apparent.

§ 116. *Every alternant with rational integral elements contains as a factor the difference-product of its variables.*

Let the variables be $a, b, \ldots\ldots, r, p, q, k, l$.

By substituting for l any of the other variables, we should cause the determinant to vanish, hence it follows that

$$l-k,\ l-q,\ l-p,\ l-r,\ \ldots\ldots,\ l-b,\ l-a$$

are factors of the determinant. Similarly

$$k-q,\ k-p,\ k-r,\ \ldots\ldots,\ k-b,\ k-a$$

are seen to be factors, and in like manner

$$\begin{array}{r} q-p,\ q-r,\ \ldots\ldots,\ q-b,\ q-a, \\ p-r,\ \ldots\ldots,\ p-b,\ p-a, \\ \ldots\ldots,\ r-b,\ r-a, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b-a. \end{array}$$

Hence if from every variable there be subtracted every variable preceding it, and the differences thus got be multiplied together, the result—known as the *difference-product* of the variables—is a factor of the determinant.

Sylvester, who uses $\zeta(a,b,c,\ldots)$ or $\zeta(abc\ldots)$ for the second power of the difference-product of $a,b,c,\ldots$, denotes the difference-product itself by $\zeta^{\frac{1}{2}}(a,b,c,\ldots)$ or $\zeta^{\frac{1}{2}}(abc\ldots)$.

§ 117. *A difference-product is itself expressible as an alternant, viz.:*

$$\zeta^{\frac{1}{2}}(a_1 a_2 a_3 \ldots\ldots a_n) = \left|\begin{array}{lllll} 1 & a_1 & a_1^2 & \ldots\ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots\ldots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \ldots\ldots & a_3^{n-1} \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ 1 & a_n & a_n^2 & \ldots\ldots & a_n^{n-1} \end{array}\right|.$$

For the alternant on the right has $\zeta^{\frac{1}{2}}(a_1 a_2 a_3 \ldots a_n)$ for a factor (§ 116), and the co-factor is readily seen, as in the particular case given at p. 41, to be unity.

§ 118. *If the expanded form of the difference-product of the* n *letters* a, b, c, ... l *be multiplied by* abc...l, *and in the result every index to a letter be made a suffix to the same letter, the expression obtained is the determinant* $|a_1 b_2 c_3 \ldots l_n|$.

This is self-evident on writing the difference-product in its determinant form.

In connection herewith see § 101.

§ 119. *The quotient of an alternant by the difference-product of its variables is a symmetric function of the variables.*

On the interchange of any pair of the variables both dividend and divisor change sign. Consequently when such an interchange is made the quotient remains unaltered, and therefore is by definition a symmetric function of its variables.

EXAMPLE:

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} (a+b+c) = \zeta^{\frac{1}{2}}(abc) \times \Sigma a.$$

§ 120. An alternant of the n^{th} order being known when we know in order its n functions and n variables, we may suitably use for it a notation in which only these $2n$ things are specified. Such a notation is obtained by taking the principal diagonal term, enclosing it in brackets, and prefixing an A. Thus

$$A\big(\phi_1(x),\ \phi_2(y),\ \phi_3(z)\big)$$

represents the alternant whose variables are x, y, z and functions ϕ_1, ϕ_2, ϕ_3, that is to say, the alternant

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) \\ \phi_1(y) & \phi_2(y) & \phi_3(y) \\ \phi_1(z) & \phi_2(z) & \phi_3(z) \end{vmatrix}.$$

The alternants requiring first to be considered are those in which the functions are *powers*, such as

$$\begin{vmatrix} a^m & a^n & a^p \\ b^m & b^n & b^p \\ c^m & c^n & c^p \end{vmatrix} \quad \text{or} \quad A(a^m b^n c^p).$$

They may be spoken of as *simple* alternants, and as they are of common occurrence, a shortening of the notation for them may be made, when there is no possibility of confusion, by leaving out the variables: thus we may use

$$\begin{aligned} & A(m,n,p) \quad \text{for} \quad A(a^m b^n c^p), \\ \text{and} \quad & A(0,1,2) \quad \text{for} \quad A(a^0 b^1 c^2) \quad \text{i.e., for} \quad \zeta^{\frac{1}{2}}(a\,b\,c). \end{aligned}$$

The form of alternant, for which the symmetric quotient referred to in § 119 is most readily obtained, and which may be viewed as the second simplest form, is that in which all the powers except the last are in order the same as those of the difference-product alternant.

§ 121. *The quotient of the alternant of the second simplest form by the corresponding difference-product is expressible as the sum of a series of fractions*, viz.:

$$\frac{A\left(a_1^0 a_2^1 \cdots a_{n-1}^{n-2} a_n^r\right)}{A\left(a_1^0 a_2^1 \cdots a_{n-1}^{n-2} a_n^{n-1}\right)} = \frac{a_n^r}{(a_n - a_{n-1})(a_n - a_{n-2}) \cdots (a_n - a_1)}$$
$$+ \frac{a_{n-1}^r}{(a_{n-1} - a_n)(a_{n-1} - a_{n-2}) \cdots (a_{n-1} - a_1)}$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \frac{a_1^r}{(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n)}.$$

Expressing

$$\begin{vmatrix} 1 & a_1^1 & a_1^2 & \cdots\cdots a_1^{n-2} & a_1^r \\ 1 & a_2^1 & a_2^2 & \cdots\cdots a_2^{n-2} & a_2^r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n-1}^1 & a_{n-1}^2 & \cdots\cdots a_{n-1}^{n-2} & a_{n-1}^r \\ 1 & a_n^1 & a_n^2 & \cdots\cdots a_n^{n-2} & a_n^r \end{vmatrix}$$

in terms of the elements of the last column and their complementaries which are all difference-products, we have

$$a_n^r \zeta^{\frac{1}{2}}(a_1 a_2 \ldots a_{n-1}) - a_{n-1}^r \zeta^{\frac{1}{2}}(a_1 a_2 \ldots a_{n-2} a_n) + a_{n-2}^r \zeta^{\frac{1}{2}}(a_1 a_2 \ldots a_{n-3} a_{n-1} a_n)$$
$$- \cdots\cdots\cdots\cdots\cdots\cdots + (-1)^{n-1} a_1^r \zeta^{\frac{1}{2}}(a_2 a_3 \ldots a_{n-1} a_n).$$

Hence dividing by the difference-product of $a_1, a_2, \ldots, a_{n-1}, a_n$, and legitimately altering the signs so as to have in the second denominator a_{n-1} in every case the minuend, in the third denominator a_{n-2}, and so on, there results the identity as stated.

§ 122. *The quotient of an alternant of the second simplest form by the corresponding difference-product is expressible* *by means of two simpler like quotients*, viz.:

$$\frac{A\left(a_1^0 a_2^1 \ldots a_{n-1}^{n-2} a_n^r\right)}{\zeta^{\frac{1}{2}}(a_1 a_2 \ldots a_{n-1} a_n)} = a_n \frac{A\left(a_1^0 a_2^1 \ldots a_{n-1}^{n-2} a_n^{r-1}\right)}{\zeta^{\frac{1}{2}}(a_1 a_2 \ldots a_{n-1} a_n)} + \frac{A\left(a_1^0 a_2^1 \ldots a_{n-2}^{n-3} a_{n-1}^{r-1}\right)}{\zeta^{\frac{1}{2}}(a_1 a_2 \ldots a_{n-2} a_{n-1})}.$$

From § 28 we have $A\left(a_1^0 a_2^1 \ldots a_{n-1}^{n-2} a_n^r\right) - a_n A\left(a_1^0 a_2^1 \ldots a_{n-1}^{n-2} a_n^{r-1}\right)$

$$= \begin{vmatrix} 1 & a_1^1 & a_1^2 & \ldots a_1^{n-2} & a_1^r - a_n a_1^{r-1} \\ 1 & a_2^1 & a_2^2 & \ldots a_2^{n-2} & a_2^r - a_n a_2^{r-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n-1}^1 & a_{n-1}^2 & \ldots a_{n-1}^{n-2} & a_{n-1}^r - a_n a_{n-1}^{r-1} \\ 1 & a_n^1 & a_n^2 & \ldots a_n^{n-2} & 0 \end{vmatrix},$$

and (§ 121) the right-hand member of this divided by $\zeta^{\frac{1}{2}}(a_1 a_2 \ldots a_n)$

$$= \left\{ \begin{array}{l} \dfrac{0}{(0-a_{n-1})(0-a_{n-2})\ldots(0-a_1)} \\ + \dfrac{a_{n-1}^r - a_n a_{n-1}^{r-1}}{(a_{n-1}-a_n)(a_{n-1}-a_{n-2})\ldots(a_{n-1}-a_1)} \\ + \cdots\cdots\cdots\cdots\cdots\cdots \\ + \dfrac{a_1^r - a_n a_1^{r-1}}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)} \end{array} \right. = \left\{ \begin{array}{l} 0 \\ + \dfrac{a_{n-1}^{r-1}}{(a_{n-1}-a_{n-2})\ldots(a_{n-1}-a_1)} \\ + \cdots\cdots\cdots\cdots\cdots\cdots \\ + \dfrac{a_1^{r-1}}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_{n-1})} \end{array} \right.$$

$$= \frac{A\left(a_1^0 a_2^1 \ldots a_{n-2}^{n-3} a_{n-1}^{r-1}\right)}{\zeta^{\frac{1}{2}}(a_1 a_2 \ldots a_{n-2} a_{n-1})} \quad (\S\ 121);$$

whence the required identity.

Example:—

$$\frac{\begin{vmatrix} 1 & a & a^4 \\ 1 & b & b^4 \\ 1 & c & c^4 \end{vmatrix}}{\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}} = c \frac{\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}}{\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}} + \frac{\begin{vmatrix} 1 & a^3 \\ 1 & b^3 \end{vmatrix}}{\begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix}},$$

$$= c(a+b+c) + \frac{b^3-a^3}{b-a}, \qquad (\S\,119,\ \text{Ex.})$$

$$= a^2+b^2+c^2+ab+bc+ca = \Sigma a^2 + \Sigma ab.$$

Similarly with the help of this result we may find the like expansion of

$$\frac{A(a^0 b^1 c^5)}{\zeta^{\frac{1}{2}}(a\,b\,c\,)},$$

thence that of

$$\frac{A(a^0 b^1 c^6)}{\zeta^{\frac{1}{2}}(a\,b\,c\,)},$$

and so on.

§ 123. *The quotient of an alternant of the second simplest form by the corresponding difference-product is equal to the sum of all the terms which can be formed by multiplying together such positive integral powers of the variables that the sum of their indices may equal the excess of the last index of the one alternant over the last index of the other.*

Let the quotient referred to be

$$\frac{A\left(a_1^0\, a_2^1 \ldots a_{n-1}^{n-2}\, a_n^r\right)}{\zeta^{\frac{1}{2}}(a_1\, a_2\, \ldots\, a_{n-1}\, a_n)}.$$

Since the dividend consists of terms like its principal diagonal term

$$a_1^0\, a_2^1\, a_3^2 \ldots a_{n-1}^{n-2}\, a_n^r,$$

and the divisor, similarly, of terms like

$$a_1^0\, a_2^1\, a_3^2 \ldots a_{n-1}^{n-2}\, a_n^{n-1},$$

we see at once that the quotient must consist of terms of the kind specified in the statement of the theorem, and of no others. It thus only remains to be shown that all possible terms of this kind occur, and that, unlike the terms of dividend and divisor, all are positive.

As an instance of such a term, viz., one the sum of whose indices is $r-n+1$, let us take

$$a_n^2 a_{n-1}^1 \cdots a_3^5 a_2^3 a_1^4 .$$

By repeated use of the theorem of § 122 we have

$$\frac{A(a_1^0 \cdots a_{n-1}^{n-2} a_n^r)}{\zeta^{\frac{1}{2}}(a_1, \cdots a_{n-1} a_n)} = \frac{A(a_1^0 \cdots a_{n-2}^{n-3} a_{n-1}^{r-1})}{\zeta^{\frac{1}{2}}(a_1 \cdots a_{n-1})} + a_n \frac{A(a_1^0 \cdots a_{n-2}^{n-3} a_{n-1}^{r-2})}{\zeta^{\frac{1}{2}}(a_1 \cdots a_{n-1})}$$
$$+ a_n^2 \frac{A(a_1^0 \cdots a_{n-2}^{n-3} a_{n-1}^{r-3})}{\zeta^{\frac{1}{2}}(a_1 \cdots a_{n-1})} + \ldots . + a_n^{r-n} \frac{A(a_1^0 \cdots a_{n-1}^{n-1})}{\zeta^{\frac{1}{2}}(a_1 \cdots a_{n-1})} + a_n^{r-n+1} ,$$

the quotient we are concerned with being thus separated into $r-n+2$ groups of terms, viz., those independent of a_n, those containing a_n^1, those containing a_n^2, and so on. Now the third, if any, of these groups must contain the term in question. Taking it, therefore, and developing the co-factor of a_n^2 according to ascending powers of a_{n-1} in the same way, we see that the co-factor of $a_n^2 a_{n-1}^1$ is

$$\frac{A(a_1^0 \cdots a_{n-3}^{n-4} a_{n-2}^{r-5})}{\zeta^{\frac{1}{2}}(a_1 \cdots a_{n-3} a_{n-2})} .$$

Expanding this co-efficient in like manner, and so on, we shall at length come to the coefficient of $a_n^2 a_{n-1}^1 \cdots a_3^5$, which, as it must be of the seventh degree and must contain only a_2 and a_1, must be

$$\frac{A(a_1^0 a_2^8)}{\zeta^{\frac{1}{2}}(a_1 a_2)} .$$

This, however,

$$= \frac{a_2^8 - a_1^8}{a_2 - a_1}$$

$$= a_2^7 + \ldots + a_2^3 a_1^4 + \ldots + a_1^7 .$$

Thus $+\ a_n^2 a_{n-1}^1 \dots a_3^5 a_2^3 a_1^4$ is a term of the expansion, and evidently the same may be proved in regard to any other term of the kind.

The quotient specified in this theorem is known as the "*complete symmetric function of* $a_1, a_2, a_3, \dots, a_n$ *of the degree* r−n+1," and is denoted by $(a_1, a_2, a_3, \dots, a_n)^{r-n+1}$, or, when there is no doubt as to the variables, by $(\ )^{r-n+1}$ or H_{r-n+1}. The theorem thus simply is

$$\frac{A\left(a_1^0 a_2^1 \dots a_{n-1}^{n-2} a_n^r\right)}{A\left(a_1^0 a_2^1 \dots a_{n-1}^{n-2} a_n^{n-1}\right)} \text{ or } \frac{A\left(a_1^0 \dots a_{n-1}^{n-2} a_n^r\right)}{\zeta^{\frac{1}{2}}\left(a_1 \dots a_{n-1} a_n\right)} = (a_1, a_2, a_3, \dots, a_n)^{r-n+1}.$$

EXAMPLES:—

$$\begin{vmatrix} 1 & a & a^5 \\ 1 & b & b^5 \\ 1 & c & c^5 \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a, b, c)^{5-2},$$

$$= \Sigma a^3 + \Sigma a^2 b + \Sigma abc,$$

$$= (a^3 + b^3 + c^3) + (a^2 b + a^2 c + b^2 a + b^2 c + c^2 a + c^2 b) + abc.$$

and

$$\frac{A(a^0 b^1 c^2 d^7)}{A(a^0 b^1 c^2 d^3)} = (a, b, c, d)^{7-3},$$

$$= \Sigma a^4 + \Sigma a^3 b + \Sigma a^2 b^2 + \Sigma a^2 bc + \Sigma abcd,$$

$$= (a^4 + b^4 + c^4 + d^4) + \dots.$$

§ 124. From the preceding it follows that every known theorem regarding complete symmetric functions becomes a theorem in alternants. Of these known theorems the simplest flow readily from the law of development

$$(a_1 a_2 \dots a_n)^s = a_n (a_1 a_2 \dots a_n)^{s-1} + (a_1 a_2 \dots a_{n-1})^s \qquad (1)$$

established in § 122 in regard to the functions in their alternant form.

Thus, developing the first term of the right-hand member of (1) by means of (1) itself, we have

$$(a_1 \dots a_n)^s = a_n^s + a_n^{s-1}(a_1 \dots a_{n-1})^1 + a_n^{s-2}(a_1 \dots a_{n-1})^2 + \dots + (a_1 \dots a_{n-1})^s, \quad (2)$$

as has already been incidentally seen in § 123. By development of the *second* term of the right-hand member in the same way, there results

$$(a_1\ldots a_n)^s = a_n(a_1\ldots a_n)^{s-1} + a_{n-1}(a_1\ldots a_{n-1})^{s-1} + a_{n-2}(a_1\ldots a_{n-2})^{s-1} + \ldots + a_1^s . \quad (3)$$

By developing *both* terms, and making a third use of (1) to combine two of the terms resulting, we have

$$(a_1\ldots a_n)^s = (a_1\ldots a_{n-2})^s + (a_{n-1}a_n)^1(a_1\ldots a_{n-1})^{s-1} + a_n^2(a_1\ldots a_n)^{s-2},$$

and thence

$$(a_1\ldots a_n)^s = (a_1\ldots a_{n-3})^s + (a_{n-2}a_{n-1}a_n)^1(a_1\ldots a_{n-2})^{s-1} + (a_{n-1}a_n)^2(a_1\ldots a_{n-1})^{s-2} + a_n^3(a_1\ldots a_n)^{s-3},$$

and, finally,

$$(a_1\ldots a_n)^s = a_1^s + (a_2\ldots a_n)^1(a_1a_2)^{s-1} + (a_3\ldots a_n)^2(a_1a_2a_3)^{s-2} + \ldots\ldots + a_n^{n-1}(a_1\ldots a_n)^{s-n+1}. \quad (4)$$

Again, returning to the first term of the right-hand member of (1), and altering it by means of (1) in another way, we have

$$(a_1\ldots a_n)^s = a_n\left\{(a_1\ldots a_{n+1})^{s-1} - a_{n+1}(a_1\ldots a_{n+1})^{s-2}\right\} + (a_1\ldots a_{n-1})^s;$$

Similarly

$$(a_1\ldots a_{n-1}a_{n+1})^s = a_{n+1}\left\{(a_1\ldots a_{n+1})^{s-1} - a_n(a_1\ldots a_{n+1})^{s-2}\right\} + (a_1\ldots a_{n-1})^s;$$

and therefore by subtraction and division

$$\frac{(a_1\ldots a_{n-1}a_n)^s - (a_1\ldots a_{n-1}a_{n+1})^s}{a_n - a_{n+1}} = (a_1\ldots a_n a_{n+1})^{s-1}. \quad (5)$$

§ 125. *The quotient of any simple alternant by the corresponding difference-product is expressible as a determinant whose elements are complete symmetric functions of the variables,* viz.:

$$\frac{A(a_1^p a^q \dots a_n^s)}{\zeta^{\frac{1}{2}}(a_1 a_2 \dots a_n)} = \begin{vmatrix} (a_1 \dots a_n)^p & (a_1 \dots a_n)^q & \dots\dots & (a_1 \dots a_n)^s \\ (a_1 \dots a_n)^{p-1} & (a_1 \dots a_n)^{q-1} & \dots\dots & (a_1 \dots a_n)^{s-1} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ (a_1 \dots a_n)^{p-n+1} & (a_1 \dots a_n)^{q-n+1} & \dots\dots & (a_1 \dots a_n)^{s-n+1} \end{vmatrix}.$$

Subtracting each element of the first row of the given alternant from the corresponding element of all the following rows, we see that the factors $a_2 - a_1$, $a_3 - a_1$, ..., $a_n - a_1$ may be taken out, and that this being done the resulting determinant is

$$\begin{vmatrix} a_1^p & a_1^q & \dots\dots & a_1^s \\ (a_1 a_2)^{p-1} & (a_1 a_2)^{q-1} & \dots\dots & (a_1 a_2)^{s-1} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ (a_1 a_n)^{p-1} & (a_1 a_n)^{q-1} & \dots\dots & (a_1 a_n)^{s-1} \end{vmatrix}.$$

Treating this in the same way, the elements of the second row being now the subtrahends, we can (§ 124 (5)) remove the factors $a_3 - a_2$, $a_4 - a_2$, ..., $(a_n - a_2)$; and continuing the process we find finally

$$\frac{A(a_1^p a_2^q \dots a_n^s)}{\zeta^{\frac{1}{2}}(a_1 a_2 \dots a_n)} = \begin{vmatrix} a_1^p & a_1^q & \dots\dots & a_1^s \\ (a_1 a_2)^{p-1} & (a_1 a_2)^{q-1} & \dots\dots & (a_1 a_2)^{s-1} \\ (a_1 a_2 a_3)^{p-2} & (a_1 a_2 a_3)^{q-2} & \dots\dots & (a_1 a_2 a_3)^{s-2} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ (a_1 \dots a_n)^{p-n+1} & (a_1 \dots a_n)^{q-n+1} & \dots\dots & (a_1 \dots a_n)^{s-n+1} \end{vmatrix}.$$

Multiplying columnwise the right-hand member of this by unity in the form

$$\begin{vmatrix} 1 & 0 & 0 & \dots\dots & 0 \\ (a_2 \dots a_n)^1 & 1 & 0 & \dots\dots & 0 \\ (a_3 \dots a_n)^2 & (a_3 \dots a_n)^1 & 1 & \dots\dots & 0 \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_n^{n-1} & a_n^{n-2} & a_n^{n-3} & \dots\dots & 1 \end{vmatrix},$$

and using (4) of § 124 we obtain the result required.

If p be other than 0 it is better to remove the factor $a_1^p a_2^p \ldots a_n^p$ before applying the theorem.

EXAMPLE:—

$$\begin{vmatrix} 1 & a^3 & a^4 \\ 1 & b^3 & b^4 \\ 1 & c^3 & c^4 \end{vmatrix} = \zeta^{\frac{1}{2}}(a\,b\,c) \begin{vmatrix} (a, b, c)^0 & (a, b, c)^3 & (a, b, c)^4 \\ 0 & (a, b, c)^2 & (a, b, c)^3 \\ 0 & (a, b, c)^1 & (a, b, c)^2 \end{vmatrix},$$

$$= \zeta^{\frac{1}{2}}(a\,b\,c) \begin{vmatrix} \Sigma a^2 + \Sigma ab & \Sigma a^3 + \Sigma a^2 b + \Sigma abc \\ \Sigma a & \Sigma a^2 + \Sigma ab \end{vmatrix},$$

$$= \zeta^{\frac{1}{2}}(a\,b\,c) \begin{vmatrix} -\Sigma ab & -\Sigma a^2 b - 2\Sigma abc \\ \Sigma a & \Sigma a^2 + \Sigma ab \end{vmatrix} = \zeta^{\frac{1}{2}}(abc) \begin{vmatrix} -\Sigma ab & \Sigma abc \\ \Sigma a & -\Sigma ab \end{vmatrix},$$

$$= \zeta^{\frac{1}{2}}(a\,b\,c) \times (\Sigma a^2 b + \Sigma abc).$$

§ 126. An alternant in which the elements are polynomial may often be expressed in terms of a simple alternant. This is evident when we consider that the converse is true, viz., that the mere transformation of a simple alternant or the multiplication of it by a non-alternant expression may lead to an alternant with polynomial elements. For example, we obtain an alternant with polynomial elements if we multiply $A(a_1^p a_1^q \ldots a_n^s)$ row-wise by $|a_{1n}|$; the multiplication column-wise gives of course the same alternating function, but the result is not an alternant in form.

EXAMPLE:—

$$\begin{vmatrix} \lambda_1 a^3 + \mu_1 a^2 + \nu_1 & \lambda_2 a^2 + \mu_2 & 1 \\ \lambda_1 b^3 + \mu_1 b^2 + \nu_1 & \lambda_2 b^2 + \mu_2 & 1 \\ \lambda_1 c^3 + \mu_1 c^2 + \nu_1 & \lambda_2 c^2 + \mu_2 & 1 \end{vmatrix} = \lambda_2 \begin{vmatrix} \lambda_1 a^3 & a^2 & 1 \\ \lambda_1 b^3 & b^2 & 1 \\ \lambda_1 c^3 & c^2 & 1 \end{vmatrix}, \qquad (\S\ 32)$$

$$= -\lambda_1 \lambda_2 \zeta^{\frac{1}{2}}(a\,b\,c) \begin{vmatrix} (\ )^0 & (\ )^2 & (\ )^3 \\ 0 & (\ )^1 & (\ )^2 \\ 0 & (\ )^0 & (\ )^1 \end{vmatrix} = -\lambda_1 \lambda_2 \zeta^{\frac{1}{2}}(abc) \begin{vmatrix} \Sigma a & \Sigma a^2 + \Sigma ab \\ 1 & \Sigma a \end{vmatrix},$$

$$= -\lambda_1 \lambda_2 \zeta^{\frac{1}{2}}(a\,b\,c) \times \Sigma ab.$$

§ 127. *In an alternant with rational integral elements the co-factor of the difference-product is expressible as a determinant whose elements are* (1) *the co-efficients in the elements of the alternant,* (2) *those symmetric functions of the variables which are linear with respect to each variable, viz.:*

$$\frac{A\left\{f_1(a_1)f_2(a_2)\dots f_n(a_n)\right\}}{\zeta^{\frac{1}{2}}(a_1a_2\dots a_n)}=\begin{vmatrix} c_{01} & c_{11} & c_{21} & \cdots & c_{n1} & c_{n+1,1} & \cdots & c_{r-1,1} & c_{r1} \\ c_{02} & c_{12} & c_{22} & \cdots & c_{n2} & c_{n+1,2} & \cdots & c_{r-1,2} & c_{r2} \\ \hdotsfor{9} \\ c_{0n} & c_{1n} & c_{2n} & \cdots & c_{nn} & c_{n+1,n} & \cdots & c_{r-1,n} & c_{rn} \\ C_n & C_{n-1} & C_{n-2} & \cdots & C_0 & 0 & \cdots & 0 & 0 \\ 0 & C_n & C_{n-1} & \cdots & C_1 & C_0 & \cdots & 0 & 0 \\ 0 & 0 & C_n & \cdots & C_2 & C_1 & \cdots & 0 & 0 \\ \hdotsfor{9} \\ 0 & 0 & 0 & \cdots & C_{r-n} & C_{r-n-1} & \cdots & C_1 & C_0 \end{vmatrix},$$

where $$f_s(x)=c_{0s}+c_{1s}x+c_{2s}x^2+\dots+c_{rs}x^r,$$

and $C_0=1,\ C_1=-\Sigma a_1,\ C_2=\Sigma a_1a_2,\ C_3=-\Sigma a_1a_2a_3, \dots$

Denoting the right-hand member by Δ, and multiplying it row-wise by $\zeta^{\frac{1}{2}}(a_1\dots a_n\omega_0\omega_1\dots\omega_{r-n})$, we have

$$\begin{vmatrix} f_1(a_1) & f_2(a_1) & \cdots & f_n(a_1) & \phi(a_1) & a_1\phi(a_1) & \cdots & a_1^{r-n}\phi(a_1) \\ f_1(a_2) & f_2(a_2) & \cdots & f_n(a_2) & \phi(a_2) & a_2\phi(a_2) & \cdots & a_2^{r-n}\phi(a_2) \\ \hdotsfor{8} \\ f_1(a_n) & f_2(a_n) & \cdots & f_n(a_n) & \phi(a_n) & a_n\phi(a_n) & \cdots & a_n^{r-n}\phi(a_n) \\ f_1(\omega_0) & f_2(\omega_0) & \cdots & f_n(\omega_0) & \phi(\omega_0) & \omega_0\phi(\omega_0) & \cdots & \omega_0^{r-n}\phi(\omega_0) \\ f_1(\omega_1) & f_2(\omega_1) & \cdots & f_n(\omega_1) & \phi(\omega_1) & \omega_1\phi(\omega_1) & \cdots & \omega_1^{r-n}\phi(\omega_1) \\ \hdotsfor{8} \\ f_1(\omega_{r-n}) & f_2(\omega_{r-n}) & \cdots & f_n(\omega_{r-n}) & \phi(\omega_{r-n}) & \omega_{r-n}\phi(\omega_{r-n}) & \cdots & \omega_{r-n}^{r-n}\phi(\omega_{r-n}) \end{vmatrix},$$

where for shortness we put

$$x^n + C_1x^{n-1} + \ldots + C_n \quad \text{or} \quad (x-a_1)(x-a_2)\ldots(x-a_n) = \phi(x).$$

In this result, however, $\phi(a_1) = 0 = \phi(a_2) = \ldots = \phi(a_n)$, hence (§ 79) it may be retransformed into a product, viz.:

$$\begin{vmatrix} f_1(a_1) & f_2(a_1) \ldots\ldots f_n(a_1) \\ f_1(a_2) & f_2(a_2) \ldots\ldots f_n(a_2) \\ & \\ f_1(a_n) & f_2(a_n) \ldots\ldots f_n(a_n) \end{vmatrix} \times \begin{vmatrix} \phi(\omega_0) & \omega_0\phi(\omega_0) & \ldots\ldots\omega_0^{r-n}\phi(\omega_0) \\ \phi(\omega_1) & \omega_1\phi(\omega_1) & \ldots\ldots\omega_1^{r-n}\phi(\omega_1) \\ & & \\ \phi(\omega_{r-n}) & \omega_{r-n}\phi(\omega_{r-n}) & \ldots\ldots\omega_{r-n}^{r-n}\phi(\omega_{r-n}) \end{vmatrix},$$

and therefore (§§ 26, 117) into

$$A\Big\{f_1(a_1)\,f_2(a_2)\ldots f_n(a_n)\Big\} \times \phi(\omega_0)\,\phi(\omega_1)\ldots\phi(\omega_{r-n}) \\ \times \zeta^{\frac{1}{2}}(\omega_0,\,\omega_1\ldots\omega_{r-n}).$$

Putting now the original from which this came, viz.:

$$\Delta\zeta^{\frac{1}{2}}(a_1\ldots a_n\omega_0\omega_1\ldots\omega_{r-n})$$

into the form

$$\Delta\zeta^{\frac{1}{2}}(a_1\ldots a_n)\,\zeta^{\frac{1}{2}}(\omega_0\omega_1\ldots\omega_{r-n})\,\phi(\omega_0)\,\phi(\omega_1)\ldots\phi(\omega_{r-n}),$$

and removing the common factors, we have

$$A\Big\{f_1(a_1)\,f_2(a_2)\ldots f_n(a_n)\Big\} = \zeta^{\frac{1}{2}}(a_1,\,a_2,\,\ldots,\,a_n)\times\Delta,$$

as was to be proved.

Example:— Taking $f_1(x) = \nu_1 + \mu_1x^2 + \lambda_1x^3,$

$f_2(x) = \mu_2 + \lambda_2x^2,$

$f_3(x) = 1;$

then
$$\frac{A\{f_1(a)\,f_2(b)\,f_3(c)\}}{\zeta^{\frac{1}{2}}(a,\,b,\,c)} = \begin{vmatrix} \nu_1 & 0 & \mu_1 & \lambda_1 \\ \mu_2 & 0 & \lambda_2 & 0 \\ 1 & 0 & 0 & 0 \\ C_3 & C_2 & C_1 & C_0 \end{vmatrix} = -\lambda_1\lambda_2C_2 = -\lambda_1\lambda_2\Sigma ab,$$

as we have already found (§ 126, Ex.).

§ 128. The foregoing theorem, though evidently much more general than that of § 125, does not include the latter as a particular case. There the result is obtained in terms of *complete* symmetric functions of the variables, here in terms of *single* symmetric functions. Applying the general theorem to the case considered in § 125, viz., where

$$f_1(x) = x^p,\ f_2(x) = x^q,\ \ldots.,\ f_n(x) = x^z,$$

we see that in Δ the elements of the first n rows are all 0 except the element in the first row and $(p+1)^{\text{th}}$ column, the element in the second row and $(q+1)^{\text{th}}$ column, and so on up to and including the element in the z^{th} row and $(z+1)^{\text{th}}$ column, the excepted elements being 1. Consequently Δ may be reduced to a determinant Δ' of the $(z+1-n)^{\text{th}}$ order with the sign-factor

$$(-1)^{p+q+\ldots+z+\frac{1}{2}n(n-1)},$$

the columns thrown out from Δ being the $(p+1)^{\text{th}}$, $(q+1)^{\text{th}}$, &c., or, what is the same thing, the columns ending with C_{z-p}, C_{z-q}, &c. Hence the last row of Δ' will contain all the C's except these, and as in any of the other rows the suffix of any C is less by unity than that of the C below it, Δ' is thus fully determined.

Example:—Taking the alternant $A(a^0b^3c^4)$, used in exemplifying § 125, we see that the omitted indices are 2, 1, and that the excesses of the highest index over these are 2, 3, &c. Thus the required quotient

$$= (-1)^{3+4+\frac{1}{2}(3-1)3}\begin{vmatrix} C_2 & C_1 \\ C_3 & C_2 \end{vmatrix},$$

$$= \begin{vmatrix} \Sigma ab & -\Sigma a \\ -\Sigma abc & \Sigma ab \end{vmatrix},$$

$$= \Sigma a^2b^2 + \Sigma a^2bc.$$

The final result is thus obtained more simply than before. The advantage, however, is not always on the same side. The determinant in the first case is of the n^{th} order, in the second case it is of the $(z+1-n)^{\text{th}}$ order, and of course either order may be higher than the other.

§ 129. *The product of a simple alternant and a single symmetric function of its variables is expressible as a sum of simple alternants, whose indices are got by arranging the variables in every term of the symmetric function in the same order and adding the indices of each term to the indices of the original alternant, the first to the first, the second to the second, and so on.*

Let $A(a^p b^q c^h \ldots.)$ be the alternant and $\Sigma a^\mu b^\nu c^\sigma \ldots.$ the symmetric function, the number of the variables $a, b, c, \ldots.$ being n.

From the definitions of an alternating and a symmetric function it is at once clear that their product is an alternating function. Consequently, since $a^p b^q c^h \ldots.$ is here a term of the one factor, $a^\mu b^\nu c^\sigma \ldots.$ a term of the other, and therefore $a^{p+\mu} b^{q+\nu} c^{h+\sigma} \ldots.$ a term of the product, there must occur in the product all the other terms of this type; that is to say, the alternant $A(a^{p+\mu} b^{q+\nu} c^{h+\sigma} \ldots.)$ is part of the product. Taking thus in succession all the $n!$ terms of $\Sigma a^\mu b^\nu c^\sigma \ldots.$ we have part of the product proved to be the sum of $n!$ alternants. But $n!$ alternants of the n^{th} order have $(n!)^2$ terms; and the product cannot contain more than $(n!)^2$ terms, for the number in each of the two factors is $n!$; therefore the sum of the $n!$ alternants is equal to the product.

EXAMPLE:—

$$
\begin{aligned}
A(a^0b^1c^2) \times \Sigma a^4 b &= A(a^0b^1c^2)\left\{a^4b^1c^0 + a^4b^0c^1 + a^1b^4c^0 + a^0b^4c^1 + a^1b^0c^4 + a^0b^1c^4\right\}, \\
&= A(012)\left\{(410) + (401) + (140) + (041) + (104) + (014)\right\}, \text{ say:} \\
&= A(422) + A(413) + A(152) + A(053) + A(116) + A(026), \\
&= A(134) - A(125) + A(035) + A(026). \qquad \text{§§ 27, 34.}
\end{aligned}
$$

§ 130. From this law of multiplication we naturally proceed to the corresponding division, viz., to another way of finding the quotient of a simple alternant by the corresponding difference-product.

EXAMPLE:—

$$A(a^0b^3c^4) \div A(a^0b^1c^2) = \Sigma a^2b^2 + \Sigma a^2bc \qquad \text{(as before).}$$

Process of Division.

$$\begin{array}{l|l|l}
\underline{A(012)\ } & A(034) & \underline{\ (022)+(112).} \\
 & \underline{A(034)-A(124)} & \\
 & \qquad\qquad\quad A(124) & \\
 & \qquad\qquad\quad \underline{A(124)} &
\end{array}$$

Here, subtracting the indices in the divisor from those in the dividend, we have 0, 2, 2, which gives the first term of the quotient: then we multiply, and proceed generally in the usual way of division.

§ 131. A determinant may evidently be an alternant with respect to two sets of variables, the interchange of any two of the one set being equivalent to an interchange of rows, and of any two of the other to an interchange of columns. Such a *double* alternant is of the form

$$\begin{vmatrix} F(\alpha_1\beta_1) & F(\alpha_1\beta_2) & \ldots\ldots \\ F(\alpha_2\beta_1) & F(\alpha_2\beta_2) & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} \quad \text{or} \quad \left| F(\alpha_1\beta_1), F(\alpha_2\beta_2) \ldots F(\alpha_n\beta_n) \right|$$

and is divisible when the elements are integral and rational by $\zeta^{\frac{1}{2}}(\alpha_1\alpha_2\ldots\alpha_n)\zeta^{\frac{1}{2}}(\beta_1\beta_2\ldots\beta_n)$. An example is readily obtained as in § 126 by multiplying row-wise the three determinants $|c_{0,n-1}|\zeta^{\frac{1}{2}}(\alpha_1\ldots\alpha_n)\zeta^{\frac{1}{2}}(\beta_1\ldots\beta_n)$.

EXAMPLE:—

$$\begin{vmatrix} c_{00}+c_{10}\alpha_1+c_{01}\beta_1+c_{11}\alpha_1\beta_1 & c_{00}+c_{10}\alpha_1+c_{01}\beta_2+c_{11}\alpha_1\beta_2 \\ c_{00}+c_{10}\alpha_2+c_{01}\beta_1+c_{11}\alpha_2\beta_1 & c_{00}+c_{10}\alpha_2+c_{01}\beta_2+c_{11}\alpha_2\beta_2 \end{vmatrix} = \begin{vmatrix} c_{00} & c_{10} \\ c_{01} & c_{11} \end{vmatrix} \begin{vmatrix} 1 & \alpha_1 \\ 1 & \alpha_2 \end{vmatrix} \begin{vmatrix} 1 & \beta_1 \\ 1 & \beta_2 \end{vmatrix},$$

$$\text{or} \quad \begin{vmatrix} \Sigma c_{\kappa\lambda}\alpha_1^\kappa\beta_1^\lambda & \Sigma c_{\kappa\lambda}\alpha_1^\kappa\beta_2^\lambda \\ \Sigma c_{\kappa\lambda}\alpha_2^\kappa\beta_1^\lambda & \Sigma c_{\kappa\lambda}\alpha_2^\kappa\beta_2^\lambda \end{vmatrix}_{\kappa=0,1}^{\lambda=0,1} = |c_{01}|\zeta^{\frac{1}{2}}(\alpha_1\alpha_2)\zeta^{\frac{1}{2}}(\beta_1\beta_2).$$

§ 132. *If in a double alternant the function be rational and integral with respect to both variables, the co-factor of the two difference-products is expressible as a determinant whose elements are the coefficients in the elements of the alternant, those symmetric functions of the first set of variables which are linear in respect to each of them, and the same symmetric functions of the second set,* viz.:

$$\frac{\left|F(\alpha_1\beta_1)\ \dots\ F(\alpha_n\beta_n)\right|}{\zeta^{\frac{1}{2}}(\alpha_1\dots\alpha_n)\,\zeta^{\frac{1}{2}}(\beta_1\dots\beta_n)}$$

$$= (-1)^{r+n+1}\begin{vmatrix}
c_{00} & c_{10} & \dots c_{n0} & c_{n+1,0} & \dots c_{r0} & C'_n & 0 & 0 & \dots 0 \\
c_{01} & c_{11} & \dots c_{n1} & c_{n+1,1} & \dots c_{r1} & C'_{n-1} & C'_n & 0 & \dots 0 \\
\hdotsfor{9} \\
c_{0n} & c_{1n} & \dots c_{nn} & c_{n+1,n} & \dots c_{rn} & C'_0 & C'_1 & C'_2 & \dots 0 \\
\hdotsfor{9} \\
c_{0r} & c_{1r} & \dots c_{nr} & c_{n+1,r} & \dots c_{rr} & 0 & 0 & 0 & \dots C'_0 \\
C_n & C_{n-1}\dots & C_0 & 0 & \dots 0 & 0 & 0 & 0 & \dots 0 \\
0 & C_n & \dots C_1 & C_0 & \dots 0 & 0 & 0 & 0 & \dots 0 \\
0 & 0 & \dots C_2 & C_1 & \dots 0 & 0 & 0 & 0 & \dots 0 \\
\hdotsfor{9} \\
0 & 0 & \dots 0 & C_{r-n+1}\cdot\cdot & C_0 & 0 & 0 & 0 & \dots 0
\end{vmatrix},$$

where $F(xy) = \sum(c_{\kappa\lambda}x^{\kappa}y^{\lambda})$ $(\kappa=0,1,\dots,n \quad \lambda=0,1,\dots,n)$

$$C_0=1,\quad C_1=-\Sigma\alpha_1,\quad C_2=\Sigma\alpha_1\alpha_2,\quad \dots\dots$$

$$\text{and}\quad C'_0=1,\quad C'_1=-\Sigma\beta_1,\quad C'_2=\Sigma\beta_1\beta_2,\ \dots\dots$$

The proof of this is exactly similar to that in § 127, the starting point being the multiplication of the right-hand member by the *two* determinants

$$\zeta^{\frac{1}{2}}(\alpha_1\alpha_2\dots\alpha_n\omega_0\omega_1\dots\omega_{r-n}),\ \zeta^{\frac{1}{2}}(\beta_1\beta_2\dots\beta_n\pi_0\pi_1\dots\pi_{r-n})$$

in succession.

§ 133. When the elements of an alternant are *fractions* with the variables occurring in the denominators, it will generally be found suitable to clear of fractions and then apply one of the theorems already given. The same theorems suffice for the treatment of certain classes of alternants with transcendental elements, viz. those where the transcendentals are *sin*, *cos*, …, *sinh*, *cosh*, …, the expressions for these in exponentials being substituted instead. In alternants of this kind the product of the sines of the halved differences of the variables often makes its appearance as a factor.

EXERCISES. SET XV.

1. Prove that

$$\begin{vmatrix} 1 & x_2+x_3 & x_2x_3 \\ 1 & x_3+x_1 & x_3x_1 \\ 1 & x_1+x_2 & x_1x_2 \end{vmatrix} = -\zeta^{\frac{1}{2}}(x_1x_2x_3);$$

and give the corresponding expression for $\zeta^{\frac{1}{2}}(x_1x_2x_3x_4)$.

Perform the following divisions, giving the quotients in the ordinary algebraic notation :—

2. $\begin{vmatrix} a & a^2 & a^6 \\ b & b^2 & b^6 \\ c & c^2 & c^6 \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$ 3. $\begin{vmatrix} a & a^3 & a^5 \\ b & b^3 & b^5 \\ c & c^3 & c^5 \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$

Find the like expansions for

4. $|a^0b^3c^5| \div |a^0b^1c^2|$. 5. $|a^0b^4c^5| \div |a^0b^1c^2|$.
6. $|a^0b^2c^6| \div |a^0b^1c^2|$. 7. $|a^0b^1c^5d^6| \div |a^0b^1c^2d^3|$.

where $|a^mb^nc^p|$ is written for $A(a^mb^nc^p)$.

8. Express $|a^0b^mc^nd^p|\Sigma a$ as a sum of alternants.

Prove the following identities, and generalize them in such a way as to leave the right-hand members unaltered in form :—

9. $|a^0b^1c^3d^4| \div |a^0b^1c^2d^3| = \Sigma ab.$
10. $|a^0b^1c^3d^5| \div |a^0b^1c^2d^3| = \Sigma a^2b + 2\Sigma abc.$
11. $|a^0b^2c^3d^5| \div |a^0b^1c^2d^3| = \Sigma a^2bc + 3\Sigma abcd.$
12. $|a^0b^1c^4d^5| \div |a^0b^1c^2d^3| = \Sigma a^2b^2 + \Sigma a^2bc + 2\Sigma abcd.$
13. $|a^0b^1c^3d^6| \div |a^0b^1c^2d^3| = \Sigma a^3b + \Sigma a^2b^2 + 2\Sigma a^2bc + 3\Sigma abcd.$

14. Prove that

$$|a^0b^mc^n| - |a^0b^mc^{n-1}|\Sigma a + |a^0b^mc^{n-2}|\Sigma ab - |a^0b^mc^{n-3}|abc = 0.$$

15. Prove that

$$|a^0b^1c^2d^7| \div |a^0b^1c^2d^3| = \begin{vmatrix} \Sigma a & -\Sigma a^2 & \Sigma a^3 & -\Sigma a^4 \\ 1 & \Sigma a & -\Sigma a^2 & \Sigma a^3 \\ 0 & 2 & \Sigma a & -\Sigma a^2 \\ 0 & 0 & 3 & \Sigma a \end{vmatrix}.$$

16. Express $\left\{3|a^0b^1c^2d^5| - 5|a^0b^1c^3d^4|\right\} \div |a^0b^1c^2d^3|$ as a sum of second powers.

17. Prove that

$$\begin{vmatrix} \cos\frac{1}{2}(a-b) & \cos\frac{1}{2}(b-c) & \cos\frac{1}{2}(c-a) \\ \cos\frac{1}{2}(a+b) & \cos\frac{1}{2}(b+c) & \cos\frac{1}{2}(c+a) \\ \sin\frac{1}{2}(a+b) & \sin\frac{1}{2}(b+c) & \sin\frac{1}{2}(c+a) \end{vmatrix} = 2\sin\tfrac{1}{2}(a-b)\sin\tfrac{1}{2}(b-c)\sin\tfrac{1}{2}(c-a).$$

18. Show that $\left\{|a^0b^1c^2d^6| + 2|a^0b^1c^3d^5| + |a^0b^2c^3d^4|\right\} \div |a^0b^1c^2d^3|$ is expressible as a third power.

19. Find the limit of $|a^0b^2c^3d^4| \div |a^0b^1c^2d^3|$ when $a = b = c = d = 1$.

20. Prove that

$$\left\{|a^0b^2c^3| + |a^0b^1c^4|\right\}|a^0b^1c^2| = |a^0b^1c^3|^2,$$

$$\left\{|a^0b^3c^4| + |a^0b^2c^5| + |a^0b^1c^6|\right\}|a^0b^1c^2| = |a^0b^1c^4|^2;$$

and find a general identity including these.

21. Prove that

$$|a_1^0a_2^1 \dots a_{n-h}^{n-h-1}a_{n-h+1}^{n-h+1}a_{n-h+2}^{n-h+2} \dots a_n^n| = |a_1^0a_2^1 \dots a_n^{n-1}|\Sigma a_1a_2 \dots a_h.$$

22. Prove the identities

$$\left\{|a^0b^2c^4| + |a^0b^1c^5|\right\}|a^0b^1c^2| = |a^0b^1c^3||a^0b^1c^4|,$$

$$\left\{|a^0b^3c^5| + |a^0b^2c^6| + |a^0b^1c^7|\right\}|a^0b^1c^2| = |a^0b^1c^4||a^0b^1c^5|;$$

and find a general identity including them.

23. Prove that if $f_r(a) = a_r + B_ra_{r-1} + \dots + Z_r$

$$\begin{vmatrix} 1 & f_1(a) & \dots & f_{n-1}(a) \\ 1 & f_1(b) & \dots & f_{n-1}(b) \\ \dots & \dots & \dots & \dots \\ 1 & f_1(l) & \dots & f_{n-1}(l) \end{vmatrix} = \zeta^{\frac{1}{2}}(ab \dots l).$$

If the coefficient of a^r in $f_r(a)$ were A_r, what would the right-hand member require to be?

24. Prove the identities

$$\left\{|a^0b^1c^3d^4e^6| + |a^0b^2c^3d^4e^5|\right\}|a^0b^1c^2d^3e^4| = |a^0b^1c^2d^3e^5||a^0b^1c^3d^4e^5|,$$

$$\left\{|a^0b^1c^3d^4e^7| + |a^0b^2c^3d^4e^6| + |a^0b^1c^3d^5e^6|\right\}|a^0b^1c^2d^3e^4| = |a^0b^1c^2d^3e^5||a^0b^1c^3d^4e^6|;$$

and give the general identity including them.

25. Denoting by $C_{r,s}$ the coefficient of x^s in the expansion of $(1+x)^r$, find the co-factor of $\zeta^{\frac{1}{2}}(ab\ldots l)$ in

$$\begin{vmatrix} 1 & C_{a,1} & \ldots\ldots & C_{a,n-1} \\ 1 & C_{b,1} & \ldots\ldots & C_{b,n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & C_{l,1} & \ldots\ldots & C_{l,n-1} \end{vmatrix}.$$

26. Prove that

$$\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial b}+\frac{\partial}{\partial c}+\frac{\partial}{\partial d}\right)|a^m b^n c^p d^q| = m|a^{m-1}b^n c^p d^q| + n|a^m b^{n-1} c^p d^q|$$
$$+p|a^m b^n c^{p-1} d^q| + q|a^m b^n c^p d^{q-1}|.$$

For what values of m, n, p, q does this vanish, supposing $m<n<p<q$?

27. Find the co-factor of $\zeta^{\frac{1}{2}}(x_1\ldots x_n)\zeta^{\frac{1}{2}}(y_1\ldots y_n)$ in

$$\begin{vmatrix} (x_1-y_1)^{n-1} & (x_1-y_2)^{n-1} & \ldots & (x_1-y_n)^{n-1} \\ (x_2-y_1)^{n-1} & (x_2-y_2)^{n-1} & \ldots & (x_2-y_n)^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ (x_n-y_1)^{n-1} & (x_n-y_2)^{n-1} & \ldots & (x_n-y_n)^{n-1} \end{vmatrix}.$$

28. Prove that

$$\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial b}+\frac{\partial}{\partial c}+\frac{\partial}{\partial d}\right)|a^0 b^1 c^3 d^4| = \left(a^2\frac{\partial}{\partial a}+b^2\frac{\partial}{\partial b}+c^2\frac{\partial}{\partial c}+d^2\frac{\partial}{\partial d}\right)|a^0 b^1 c^2 d^3|.$$

29. Prove that

$$\left|\cos na_0,\ \cos(n-1)a_1,\ \ldots,\ \cos(0)a_n\right| \div \left|\cos^n a_0,\ \cos^{n-1}a_1,\ \ldots\ \cos^0 a_n\right| = 2^{\frac{1}{2}n(n-1)}.$$

30. For what value of ω does $\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial b}+\frac{\partial}{\partial c}+\frac{\partial}{\partial d}\right)^{\omega}|a^m b^n c^p d^q|$ vanish?

31. Prove that

$$\left|\sin(n+1)a_0,\ \sin na_1,\ \ldots,\ \sin a_n\right| \div \left|\cos^n a_0,\ \cos^{n-1}a_1,\ \ldots,\ \cos^0 a_n\right|$$
$$= 2^{\frac{1}{2}n(n-1)}\sin a_0 \sin a_1 \ldots \sin a_n.$$

32. Prove that

$$\begin{vmatrix} |a^m b^1 c^2 d^3| & |a^n b^1 c^2 d^3| & |a^p b^1 c^2 d^3| & |a^q b^1 c^2 d^3| \\ |a^0 b^m c^2 d^3| & |a^0 b^n c^2 d^3| & |a^0 b^p c^2 d^3| & |a^0 b^q c^2 d^3| \\ |a^0 b^1 c^m d^3| & |a^0 b^1 c^n d^3| & |a^0 b^1 c^p d^3| & |a^0 b^1 c^q d^3| \\ |a^0 b^1 c^2 d^m| & |a^0 b^1 c^2 d^n| & |a^0 b^1 c^2 d^p| & |a^0 b^1 c^2 g^q| \end{vmatrix} = |a^0 b^1 c^2 d^3|^3 |a^m b^n c^p d^q|.$$

33. Find the co-factor of $\zeta^{\frac{1}{2}}(x_1\ldots\ldots x_n)\zeta^{\frac{1}{2}}(y_1\ldots\ldots y_n)$ in

$$\begin{vmatrix} (x_1-y_1)^{-1} & (x_1-y_2)^{-1} & \ldots\ldots & (x_1-y_n)^{-1} \\ (x_2-y_1)^{-1} & (x_2-y_2)^{-1} & \ldots\ldots & (x_2-y_n)^{-1} \\ \ldots & \ldots & \ldots & \ldots \\ (x_n-y_1)^{-1} & (x_n-y_2)^{-1} & \ldots\ldots & (x_n-y_n)^{-1} \end{vmatrix}.$$

34. Find the co-factor of the product of the sines of the halved differences of $x_1, x_2, \ldots, x_n$, in

$$\left|\begin{array}{llllll} 1 & \sin x_1 & \cos x_1 & \sin 2x_1 & \cos 2x_1 & \cdots \\ 1 & \sin x_2 & \cos x_2 & \sin 2x_2 & \cos 2x_2 & \cdots \\ \multicolumn{6}{c}{\dotfill} \\ 1 & \sin x_{2n+1} & \cos x_{2n+1} & \sin 2x_{2n+1} & \cos 2x_{2n+1} & \cdots \end{array}\right| .$$

35. Prove that

$$\left|\begin{array}{lll} (x_1-y_1)^{-2} & (x_1-y_2)^{-2} \ldots & (x_1-y_n)^{-2} \\ (x_2-y_1)^{-2} & (x_2-y_2)^{-2} \ldots & (x_2-y_n)^{-2} \\ \multicolumn{3}{c}{\dotfill} \\ (x_n-y_1)^{-2} & (x_n-y_2)^{-2} \ldots & (x_n-y_n)^{-2} \end{array}\right| = \left|\begin{array}{ll} (x_1-y_1)^{-1} \ldots & (x_1-y_n)^{-1} \\ (x_2-y_1)^{-1} \ldots & (x_2-y_n)^{-1} \\ \multicolumn{2}{c}{\dotfill} \\ (x_n-y_1)^{-1} \ldots & (x_n-y_n)^{-1} \end{array}\right| {}^{+}\!\left|\begin{array}{ll} (x_1-y_1)^{-1} \ldots & (x_1-y_n)^{-1} \\ (x_2-y_1)^{-1} \ldots & (x_1-y_n)^{-1} \\ \multicolumn{2}{c}{\dotfill} \\ (x_n-y_1)^{-1} \ldots & (x_n-y_n)^{-1} \end{array}\right|^{+} ;$$

the second factor denoting an expression whose terms are formed from the elements exactly as if it were a determinant but are all positive.

36. Find the co-factor of $\zeta^{\frac{1}{2}}(x_1 \ldots x_n)\,\zeta^{\frac{1}{2}}(y_1 \ldots x_{n+1})$ in

$$\left|\begin{array}{llll} (x_1-y_1)^{-1} & (x_1-y_2)^{-1} & \cdots & (x_1-y_{n+1})^{-1} \\ (x_2-y_1)^{-1} & (x_2-y_2)^{-1} & \cdots & (x_2-y_{n+1})^{-1} \\ \multicolumn{4}{c}{\dotfill} \\ (x_n-y_1)^{-1} & (x_n-y_2)^{-1} & \cdots & (x_n-y_{n+1})^{-1} \\ 1 & 1 & \cdots & 1 \end{array}\right| .$$

37. Find the co-factor of $\zeta^{\frac{1}{2}}(x_1 \ldots x_n)\,\zeta^{\frac{1}{2}}(y_1 \ldots y_{n+1})$ in a determinant like that of Ex. 36, but having the indices -2 instead of -1. Show that from the result and that of Ex. 36 by putting $y_{n+1}=\infty$, the identity of Ex. 35 is obtained.

SYMMETRIC DETERMINANTS.

§ 134. In any determinant two elements which occupy corresponding positions on opposite sides of the principal diagonal, and which are therefore such that the row and column numbers of the one are respectively the column and row numbers of the other, are called *conjugate* elements. In a perfectly similar way we speak of *conjugate minors;* and, going farther, a row and column bearing the same number we call *conjugate lines* of the determinant.

A minor whose principal diagonal is coincident with that of the main determinant is called a *coaxial minor.*

§ 135. A determinant is said to be SYMMETRIC with respect to a line or point in it when every two elements symmetrically situated with respect to the line or point are equal. Of the *lines* with respect to which a determinant may be symmetric two are worth taking into account, viz. the diagonals: the only *point* is the centre, that is, the intersection of the diagonals. We have thus two kinds of symmetric determinants to consider,—axi-symmetric and centro-symmetric.

§ 136. In a centro-symmetric determinant the r^{th} row reversed forms in every case the r^{th} row from the end, that is to say, the determinant is the same when read backwards as when read forwards.

§ 137. *Every centro-symmetric determinant is expressible as the product of two determinants, of the orders* $\frac{1}{2}$n, $\frac{1}{2}$n *if* n *be even, and of the orders* $\frac{1}{2}(n+1)$, $\frac{1}{2}(n-1)$ *if* n *be odd.*

Consider, first, the determinant

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_m & a_{m+1} & \cdots & a_{2m-1} & a_{2m} \\ b_1 & b_2 & \cdots & b_m & b_{m+1} & \cdots & b_{2m-1} & b_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ l_1 & l_2 & \cdots & l_m & l_{m+1} & \cdots & l_{2m-1} & l_{2m} \\ l_{2m} & l_{2m-1} & \cdots & l_{m+1} & l_m & \cdots & l_2 & l_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{2m} & b_{2m-1} & \cdots & b_{m+1} & b_m & \cdots & b_2 & b_1 \\ a_{2m} & a_{2m-1} & \cdots & a_{m+1} & a_m & \cdots & a_2 & a_1 \end{vmatrix}$$

of the $2m^{\text{th}}$ order. Increasing each element of the first row by the corresponding element of the last row, each element of the 2nd row by the corresponding element of the 2nd row from the end, and so on, we have the determinant

$$\begin{vmatrix}
a_1+a_{2m} & a_2+a_{2m-1} & \cdots & a_m+a_{m+1} & a_m+a_{m+1} & \cdots & a_2+a_{2m-1} & a_1+a_{2m} \\
b_1+b_{2m} & b_2+b_{2m-1} & \cdots & b_m+b_{m+1} & b_m+b_{m+1} & \cdots & b_2+b_{2m-1} & b_1+b_{2m} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
l_1+l_{2m} & l_2+l_{2m-1} & \cdots & l_m+l_{m+1} & l_m+l_{m+1} & \cdots & l_2+l_{2m-1} & l_1+l_{2m} \\
l_{2m} & l_{2m-1} & \cdots & l_{m+1} & l_m & \cdots & l_2 & l_1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
b_{2m} & b_{2m-1} & \cdots & b_{m+1} & b_m & \cdots & b_2 & b_1 \\
a_{2m} & a_{2m-1} & \cdots & a_{m+1} & a_m & \cdots & a_2 & a_1
\end{vmatrix};$$

which, if each element of the last column be diminished by the corresponding element of the first column, each element of the second column from the end by the corresponding element of the second column from the beginning, and so on, yields a determinant seen to be resolvable (§ 79) into

$$\begin{vmatrix}
a_1+a_{2m} & a_2+a_{2m-1} & \cdots & a_m+a_{m+1} \\
b_1+b_{2m} & b_2+b_{2m-1} & \cdots & b_m+b_{m+1} \\
\cdots & \cdots & \cdots & \cdots \\
l_1+l_{2m} & l_2+l_{2m-1} & \cdots & l_m+l_{m+1}
\end{vmatrix} \text{ and } \begin{vmatrix}
a_1-a_{2m} & a_2-a_{2m-1} & \cdots & a_m-a_{m+1} \\
b_1-b_{2m} & b_2-b_{2m-1} & \cdots & b_m-b_{m+1} \\
\cdots & \cdots & \cdots & \cdots \\
l_1-l_{2m} & l_2-l_{2m-1} & \cdots & l_m-l_{m+1}
\end{vmatrix}.$$

Similarly it may be shown that the determinant of the $(2m+1)^{\text{th}}$ order got from the above by annexing after the m^{th} column the column $a_0 b_0 \ldots l_0 l_0 \ldots b_0 a_0$ and after the m^{th} row the row $k_1 k_2 \ldots k_m k_0 k_m \ldots k_2 k_1$ is equal to

$$\begin{vmatrix}
a_1+a_{2m} & a_2+a_{2m-1} & \cdots & a_m+a_{m+1} & 2a_0 \\
b_1+b_{2m} & b_2+b_{2m-1} & \cdots & b_m+b_{m+1} & 2b_0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
l_1+l_{2m} & l_2+l_{2m-1} & \cdots & l_m+l_{m+1} & 2l_0 \\
k_1 & k_2 & \cdots & k_m & k_0
\end{vmatrix} \begin{vmatrix}
a_1-a_{2m} & a_2-a_{2m-1} & \cdots & a_m-a_{m+1} \\
b_1-b_{2m} & b_2-b_{2m-1} & \cdots & b_m-b_{m+1} \\
\cdots & \cdots & \cdots & \cdots \\
l_1-l_{2m} & l_2-l_{2m-1} & \cdots & l_m-l_{m+1}
\end{vmatrix}.$$

§ 138. *A centro-symmetric determinant of the* $2\mathrm{m}^{th}$ *order is expressible as the difference of the squares of two sums of minors of the* m^{th} *order formed from the first* m *rows.*

The product of the two factors, D and D' say, in the first case of § 137 is equal to $\{\frac{1}{2}(D+D')\}^2 - \{\frac{1}{2}(D-D')\}^2$, and when D and D' are expanded (§ 29) in terms of determinants with monomial elements, the determinants in the one expansion are in magnitude the same as those in the other: hence the theorem.

§ 139. A determinant in which every two conjugate elements are equal is an 'axi-symmetric' determinant.

'Symmetric,' however, is in common use for 'axi-symmetric,' symmetry with respect to the principal diagonal being the form which most commonly occurs.

§ 140. *In the case of an axi-symmetric determinant (1) conjugate lines are alike, (2) coaxial minors are themselves axi-symmetric, (3) conjugate minors are equal, (4) the adjugate determinant is axi-symmetric.*

§ 141. Of the general theorems whose forms are modified by the existence in the determinant of symmetry with respect to the principal diagonal the only one worth noting here is that of § 62. If the selected lines there referred to be conjugate (*v.* example p. 84) then for axi-symmetric determinants they are alike, and the development takes the form already partly exemplified on p. 143. In symbols, the change is from

$$D(a_{1n}) = a_{11}D^{(1}_{(1} - \sum_{r=2,\ldots,n}^{s=2,\ldots,n} (-1)^{r+s} a_{1r}a_{s1}D^{(1,s}_{(r,1}$$

to

$$D(a_{1n})_{a_{rs}=a_{sr}} = a_{11}D^{(1}_{(1} - \sum_{r=2,\ldots,n} a_{1r}^2 D^{(1,r}_{(1,r} - \sum_{r=2,\ldots,n}^{s>r<n+1} (-1)^{r+s} a_{1r}a_{s1}D^{(1,s}_{(r,1}.$$

§ 142. *The product of an axi-symmetric determinant by the second power of any determinant of the same order is expressible as an axi-symmetric determinant.*

Let the first determinant be $|a_{1n}|$ with $a_{rs} = a_{sr}$, the second $|\beta_{1n}|$, the product of the two $|C_{1n}|$, and $|C_{1n}||\beta_{1n}| = |\mathsf{D}_{1n}|$. Then

$$\begin{aligned}
\mathsf{D}_{rs} &= C_{s1}\beta_{r1} + C_{s2}\beta_{r2} + \dots + C_{sn}\beta_{rn}, && \S\ 67.\\
&= \left.\begin{aligned}
&(\beta_{s1}a_{11} + \beta_{s2}a_{12} + \dots + \beta_{sn}a_{1n})\beta_{r1}\\
&+(\beta_{s1}a_{21} + \beta_{s2}a_{22} + \dots + \beta_{sn}a_{2n})\beta_{r2}\\
&+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&+(\beta_{s1}a_{n1} + \beta_{s2}a_{n2} + \dots + \beta_{sn}a_{nn})\beta_{rn}
\end{aligned}\right\} && \S\ 67.\\
&= \mathsf{D}_{sr},
\end{aligned}$$

as is at once seen on changing in order the columns of terms into rows and bearing in mind that $a_{rs} = a_{sr}$. Hence $|\mathsf{D}_{1n}|$ is axi-symmetric.

§ 143. *Any power of an axi-symmetric determinant is expressible as an axi-symmetric determinant.*

This follows at once from §§ 138, 69.

§ 144. A determinant such that each line perpendicular to the principal diagonal has all its elements alike is called a PERSYMMETRIC determinant. In the persymmetric determinant of the n^{th} order

$$\begin{vmatrix}
a_1 & a_2 & a_3 & \dots & a_n\\
a_2 & a_3 & a_4 & \dots & a_{n+1}\\
a_3 & a_4 & a_5 & \dots & a_{n+2}\\
\dots & \dots & \dots & \dots & \dots\\
a_n & a_{n+1} & a_{n+2} & \dots & a_{2n-1}
\end{vmatrix}$$

there are evidently at most $2n-1$ distinct elements, viz. those of the principal diagonal and one adjacent minor diagonal. It may thus be shortly denoted by

$$P\left(a_1 \overset{a_2}{} a_3 \overset{a_4}{} \dots\ a_{2n-1}\right) \text{ or } P(a_1\, a_2 \dots a_{2n-1}).$$

§ 145. *The persymmetric determinant of* $a_1, a_2, \ldots, a_{2n-1}$, *is equal to the persymmetric determinant of* a_1, $ma_1 + a_2$, $m^2a_1 + 2ma_2 + a_3$, $m^3a_1 + 3m^2a_2 + 3ma_3 + a_4$, etc.

This follows from multiplying the determinant row-wise by

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \ldots. \\ m & 1 & 0 & 0 & 0 \ldots. \\ m^2 & 2m & 1 & 0 & 0 \ldots. \\ m^3 & 3m^2 & 3m & 1 & 0 \ldots. \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix}$$

and repeating the operation upon the product.

Indicating row-multiplication as practised in the multiplication of determinants by $\between$,—for example, writing $(a, b, c \between \alpha, \beta, \gamma)$ for $a\alpha + b\beta + c\gamma$,—the elements of the new persymmetric determinant here found are very conveniently denoted by a_1, $(a_1, a_2 \between m, 1)$, $(a_1, a_2, a_3 \between m, 1)^2$, $(a_1, a_2, a_3, a_4 \between m, 1)^3$, etc.

When $m = -1$, these elements are the first terms of the successive difference-series of $a_1, a_2, \ldots, a_{2n-1}$, so that as a special case we have the theorem—*The persymmetric determinant of* $a_1 \ldots a_{2n-1}$, *is not altered by substituting for its elements the* 0th, 1st, 2nd, …, (2n − 2)th *differences of the first of them.*

§ 146. A determinant such that any row is got from the preceding row by passing the last element over the others to the first place is called a CIRCULANT. The circulant

$$\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{vmatrix}$$

whose first row is $a_1, a_2, a_3, \ldots, a_n$, may be denoted by $C(a_1 a_2 \ldots a_n)$.

The determinant got by changing the signs of every element on one side of the principal diagonal of a circulant is called a SKEW CIRCULANT. For it the functional symbol C' may be used.

§ 147. A circulant is evidently (§ 38) a persymmetric determinant, viz.

$$C(a_1 a_2 \dots a_n) = (-1)^{\frac{1}{2}n(n-1)} P(a_1 a_2 \dots a_n a_1 a_2 \dots a_{n-1}).$$

Also evidently (§ 35)

$$C(a_1 a_2 \dots a_n) = (-1)^{n-1} C(a_2 a_3 \dots a_n a_1),$$

and (§ 24)

$$C(a_1 a_2 \dots a_n) = C(a_1 a_n a_{n-1} \dots a_2).$$

§ 148. *A circulant contains as a factor the sum of the elements of one of its rows, the co-factor being expressible as a persymmetric determinant of the next lower order, viz.*

$$\begin{aligned} C(a_1 \dots a_n) = {} & (-1)^{\frac{1}{2}n(n-1)} (a_1 + a_2 + \dots + a_n) \\ & \times P(a_1 - a_2,\ a_2 - a_3,\ \dots,\ a_n - a_1,\ a_1 - a_2,\ \dots,\ a_{n-3} - a_{n-2}). \end{aligned}$$

Adding to each element of the last column of the persymmetric determinant in § 147 the corresponding element of all the other columns, and then removing the factor $a_1 + a_2 + \dots + a_n$ we have

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & 1 \\ a_2 & a_3 & a_4 & \dots & a_n & 1 \\ a_3 & a_4 & a_5 & \dots & a_1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & a_1 & a_2 & \dots & a_{n-2} & 1 \end{vmatrix}$$

which passes into the required persymmetric determinant if each element of the first $n-1$ rows be diminished by the element immediately below it.

§ 149. *The circulant of* $\mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_n$ *is equal to the product of all expressions of the form* $\mathrm{a}_1 + \omega_r \mathrm{a}_2 + \dots + \omega_r^{n-1} \mathrm{a}_n$, *where* ω_r *is one of the* n^{th} *roots of* 1.

Increasing each element of the 1st column of the circulant by ω_r multiplied by the corresponding element of the 2nd column, ω_r^2 multiplied by the corresponding element of the 3rd column, ω_r^3 multiplied by the corresponding element of the 4th column, etc., we have an equivalent determinant in which the first column is

$$\begin{array}{l} a_1 + \omega_r a_2 + \omega_r^2 a_3 + \dots + \omega_r^{n-1} a_n \\ a_n + \omega_r a_1 + \omega_r^2 a_2 + \dots + \omega_r^{n-1} a_{n-1} \\ a_{n-1} + \omega_r a_n + \omega_r^2 a_1 + \dots + \omega_r^{n-1} a_{n-2} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_2 + \omega_r a_3 + \omega_r^2 a_4 + \dots + \omega_r^{n-1} a_1 . \end{array}$$

But the second expression here equals the first multiplied by ω_r, the third equals the first multiplied by ω_r^2, and so on; hence the first expression is a factor of the determinant. And as the product of the n factors of which it is the type contains the term $+a_1^n$ which is evidently a term of the determinant, no additional factor is required; that is to say

$$C(a_1 a_2 \dots a_n) = \prod_{r=1}^{r=n} (a_1 + \omega_r a_2 + \omega_r^2 a_3 + \dots + \omega_r^{n-1} a_n) .$$

A suggestive proof of this identity is obtained by viewing the two members of it as different forms of the resultant of the equations $a_1 + a_2 x + a_3 x^2 + \dots + a_n x^{n-1} = 0$ and $1 - x^n = 0$.

§ 150. *The product of two circulants of the same order is expressible as a circulant.*

This is a direct result of § 67. EXAMPLE:—

$$\begin{aligned} C(abc)\,C(\alpha\beta\gamma) &= C(a\alpha + b\beta + c\gamma,\ c\alpha + a\beta + b\gamma,\ b\alpha + c\beta + a\gamma), \\ &= C(a\alpha + c\beta + b\gamma,\ b\alpha + a\beta + c\gamma,\ c\alpha + b\beta + a\gamma). \end{aligned}$$

§ 151. *A circulant of the* 2n*th* *order is expressible as the product of two determinants of the* n*th* *order, a circulant and a skew circulant,* viz.

$$\begin{aligned} C(a_1 \dots a_{2n}) &= C(a_1 + a_{n+1},\ a_2 + a_{n+2},\ \dots,\ a_n + a_{2n}) \\ &\quad \times C'(a_1 - a_{n+1},\ a_2 - a_{n+2},\ \dots,\ a_n - a_{2n}). \end{aligned}$$

Reversing the order of the last n rows and then reversing the order of the last n columns, the given circulant becomes centro-symmetric; and thus (§ 137) the theorem follows.

§ 152. *A circulant of the* $2\mathrm{n}^{th}$ *order is expressible as a circulant of the* n^{th} *order, viz.* $\mathrm{C}(\mathrm{a}_1 \ldots \mathrm{a}_{2n}) = \mathrm{C}(\mathrm{X}_1 \ldots \mathrm{X}_n)$ *where* $\mathrm{X}_r = (\mathrm{a}_1, -\mathrm{a}_2, \mathrm{a}_3, \ldots, -\mathrm{a}_{2n})(\mathrm{a}_{2r-1}, \mathrm{a}_{2r-2}, \ldots, \mathrm{a}_{2r+1}, \mathrm{a}_{2r})$.

From $C(a_1 \ldots a_{2n})$ a determinant equal to $(-1)^{\frac{1}{2}n(n+1)}C(a_1 \ldots a_{2n})$ is got by placing first the odd-numbered rows in order and then the even-numbered rows in order and altering all the signs of the even-numbered columns: another equal to $(-1)^{\frac{1}{2}(n-1)n}C(a_1 \ldots a_{2n})$ is got from this by deleting the negative signs, reversing the order of the rows and then reversing the order of the elements in each row. Multiplying these determinants together and expressing the result as the product of two of its minors (§ 79), we have

$$(-1)^n C^2(a_1 \ldots a_{2n}) = (-1)^n C^2(X_1 \ldots X_n),$$

and thence the theorem required.

This and the other theorems following § 149 may be proved by using the fundamental property there established.

§ 153. To almost every one of the theorems regarding circulants there is a corresponding theorem regarding skew circulants. Thus, following the order of §§ 147-149, we have

$$\begin{aligned} & C'(a_1 \ldots a_n) \\ & = (-1)^{\frac{1}{2}n(n-1)}P(a_1, \ldots, a_n, -a_1, -a_2, \ldots, -a_{n-1}); \\ & = (-1)^n C'(a_2, \ldots, a_n, -a_1); \\ & = C'(a_1, -a_n, -a_{n-1}, \ldots, -a_2); \end{aligned}$$

$$= (-1)^{\frac{1}{2}n(n-1)}P(a_1+a_2, a_2+a_3, \ldots, a_{n-1}+a_n, a_n-a_1, -a_1-a_2, \ldots, -a_{n-3}-a_{n-2}) \\ \times (a_1 - a_2 + a_3 - \ldots + a_n), \text{ } n \text{ being odd};$$

$$= \prod_{r=1}^{r=n}(a_1 + \omega_r a_2 + \omega_r^2 a_3 + \ldots + \omega_r^{n-1}a_n), \text{ } \omega_r \text{ being an } n^{\text{th}} \text{ root of } -1;$$

and "skew circulant" may be substituted for "circulant" in § 150.

§ 154. *A skew circulant of odd order is expressible as a circulant, viz.* $C'(a_1 \dots a_{2n+1}) = C(a_1, -a_2, a_3, -a_4, \dots, a_{2n+1})$.

This is at once obtained by changing the signs of all the elements in the even-numbered rows and then in the even-numbered columns.

EXERCISES. SET XVI.

1. If a determinant be symmetric with respect to both diagonals, to which of the above-mentioned special classes does it belong?

2. Show that any even power of any determinant is expressible as an axi-symmetric determinant.

3. Establish the identity

$$4bc\begin{vmatrix} a^2+b^2+c^2 & 2ab & 2ac \\ 2ab & a^2+b^2 & bc \\ 2ac & bc & c^2+a^2 \end{vmatrix} = (b^2+c^2-a^2)\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}.$$

4. Show that the product of any two determinants of the n^{th} order is expressible as a centro-symmetric determinant of the $2n^{\text{th}}$ order.

5. Find the linear factors of

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ b & a & d & c & f & e & h & g \\ c & d & a & b & g & h & e & f \\ d & c & b & a & h & g & f & e \\ e & f & g & h & a & b & c & d \\ f & e & h & g & b & a & d & c \\ g & h & e & f & c & d & a & b \\ h & g & f & e & d & c & b & a \end{vmatrix}.$$

6. Prove that if the sum of the elements of each row of an axi-symmetric determinant vanishes, the primary minors are equal in magnitude: and state the law regarding the difference in sign.

7. Prove that

$$\begin{vmatrix} aa'-bb'-cc' & a'b+ab' & a'c+ac' \\ a'b+ab' & bb'-cc'-aa' & b'c+bc' \\ a'c+ac' & b'c+bc' & cc'-aa'-bb' \end{vmatrix} = (a^2+b^2+c^2)(a'^2+b'^2+c'^2)(aa'+bb'+cc').$$

8. Prove that if an axi-symmetric determinant vanishes the co-factor of any element is a mean proportional between the co-factors of the principal diagonal elements belonging to the row and column of the said element. What is the alternative hypothesis for the same result?

9. Express as a continuant

$$\begin{vmatrix} a+b & a & a & a \\ b & a+b & a & a \\ b & b & a+b & a \\ b & b & b & a+b \end{vmatrix}.$$

10. An axi-symmetric determinant having been multiplied by itself, prove that if the principal diagonal elements of the first factor be diminished by x, those of the second increased by x, and those of the product diminished by x^2, an identity will still subsist.

11. Prove that the altered product in the preceding exercise when expanded according to descending powers of x has its terms alternately positive and negative.

12. Prove that the determinant whose principal diagonal is $c_1, c_2, c_3, \ldots, c_n$, elements on the one side of this diagonal each equal to a, and on the other each equal to b, is equal to $\{af(b)-bf(a)\} \div (a-b)$ where $f(x)=(c_1-x)(c_2-x)\ldots(c_n-x)$.

13. Prove that, if the complementary minor of the first element of an axi-symmetric determinant be zero, the determinant is expressible as a second power.

14. Indicate in symbols the condition for $|a_{1n}|$ being persymmetric, and show that it includes the condition $a_{rs}=a_{sr}$.

15. Express in non-determinant notation

$$\begin{vmatrix} x & a & a & b & b & y \\ d & x & a & b & y & c \\ d & d & x & y & c & c \\ c & c & y & x & d & d \\ c & y & b & a & x & d \\ y & b & b & a & a & x \end{vmatrix}.$$

16. If $x_1^2+x_2^2+x_3^2=y_1^2+\ldots=z_1^2+\ldots=1$ and $x_1y_1+x_2y_2+x_3y_3=x_1z_1+\ldots=y_1z_1+\ldots=0$, prove that

$$\begin{vmatrix} P_1 & Q_{12} & Q_{13} \\ Q_{12} & P_2 & Q_{23} \\ Q_{13} & Q_{23} & P_3 \end{vmatrix} = \begin{vmatrix} a & f & e \\ f & b & d \\ e & d & c \end{vmatrix},$$

where $P_r=ax_r^2+by_r^2+cz_r^2+2dy_rz_r+2ex_rz_r+2fx_ry_r$,

and $Q_{rs}=ax_rx_s+by_ry_s+cz_rz_s+d(y_rz_s+z_ry_s)+e(x_rz_s+z_rx_s)+f(x_ry\ +y_rx_s)$.

17. Show that if $s_r = a_1^r + a_2^r + \dots + a_n^r$ and $m < n$, $P(s_0, s_1, \dots, s_{2m-2})$ is expressible as a sum of squares of alternants.

Establish the identities.

18. $C(\Sigma a - a_1, \Sigma a - a_2, \dots, \Sigma a - a_n) = (-1)^{n-1}(n-1)\,C(a_1, \dots, a_n)$.

19. $C(\tfrac{1}{2}\Sigma - a_1, \tfrac{1}{2}\Sigma a - a_2, \dots, \tfrac{1}{2}\Sigma a - a_n) = (-1)^{n-1}(\tfrac{1}{2}n - 1)\,C(a_1, \dots, a_n)$.

20. $C(a, a+d, \dots, a+\overline{n-1}d) = (-1)^{n-1}\tfrac{1}{2}(nd)^{n-1}\{2a + (n-1)d\}$.

21. $C(a, ar, \dots, ar^{n-1}) = (-1)^{n-1}a^n(r^n - 1)^{n-1}$.

22. Prove that the differential coefficient of an axi-symmetric determinant with respect to one of the elements whose row and column numbers are r and s is equal to $(-1)^{r+s}$ multiplied by twice the complementary minor of that element.

23. Prove that $P(a_1 - a_2, a_2 - a_3, \dots, a_n - a_1, a_1 - a_2, \dots, a_{n-3} - a_{n-2})$
$= P(a_3 - a_4, a_4 - a_5, \dots, a_n - a_1, a_1 - a_2, \dots, a_{n-1} - a_n)$.

24. Find the differential coefficient of $C(a_1 \dots a_n)$ with respect to a_1.

25. Find a general identity including that in Ex. 7, and thence obtain a determinant equivalent to $(a_1^2 + a_2^2 + \dots + a_n^2)^n$.

26. Show that

$$\begin{vmatrix} 1 & a_1 & a_1 a_2 & \dots & (a_1 \dots a_{n-1}) \\ 1 & a_2 & a_2 a_3 & \dots & (a_2 \dots a_n) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n a_{n+1} & \dots & (a_n \dots a_{2n-2}) \end{vmatrix}$$

is expressible as the quotient of a persymmetric determinant by

$$a_0^n a_1^{n-1} a_2^{n-2} \dots a_{n-1}.$$

27. Express in non-determinant notation the circulant whose first row consists of p negative units followed by $n-p$ positive units, p being less than $\tfrac{1}{2}n$.

28. Prove that if $D(a_{1n})_{a_{rs} = a_{sr}} = 0$, then $\left(\dfrac{\partial D}{\partial a_{rs}}\right)^2 = 4\dfrac{\partial D}{\partial a_{ss}}\dfrac{\partial D}{\partial a_{rr}}$.

29. Find the value of $P(a, ar, ar^2, \dots, ar^{2n-2})$.

30. Show that the product of a circulant and a skew circulant can be expressed as a circulant.

31. If U_n be the number of distinct terms in an axi-symmetric determinant of order n, prove that

$$U_n = nU_{n-1} - \tfrac{1}{2}(n-1)(n-2)U_{n-3},$$

and thence calculate $U_2, U_3, \dots, U_6$.

32. Prove that the persymmetric determinant of $\cos a$, $\cos(a+d)$, ..., $\cos(a + \overline{2n-1}d)$ vanishes for all values of n greater than 2.

33. Show that a circulant of the $(4n+2)^{\text{th}}$ order is expressible in two ways as a circulant of the $(2n+1)^{\text{th}}$ order.

34. Prove that

$$-|a_1\beta_2\gamma_3|^2\begin{vmatrix} a & b & c\\ b & d & e\\ c & e & f\end{vmatrix}^2 = \left\|\begin{array}{ccc}
\begin{vmatrix} a & b & c & a_1\\ b & d & e & a_2\\ c & e & f & a_3\\ a_1 & a_2 & a_3 & 0\end{vmatrix} & \begin{vmatrix} a & b & c & a_1\\ b & d & e & a_2\\ c & e & f & a_3\\ \beta_1 & \beta_2 & \beta_3 & 0\end{vmatrix} & \begin{vmatrix} a & b & c & a_1\\ b & d & e & a_2\\ c & e & f & a_3\\ \gamma_1 & \gamma_2 & \gamma_3 & 0\end{vmatrix}\\
\begin{vmatrix} a & b & c & \beta_1\\ b & d & e & \beta_2\\ c & e & f & \beta_3\\ a_1 & a_2 & a_3 & 0\end{vmatrix} & \begin{vmatrix} a & b & c & \beta_1\\ b & d & e & \beta_2\\ c & e & f & \beta_3\\ \beta_1 & \beta_2 & \beta_3 & 0\end{vmatrix} & \begin{vmatrix} a & b & c & \beta_1\\ b & d & e & \beta_2\\ c & e & f & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3 & 0\end{vmatrix}\\
\begin{vmatrix} a & b & c & \gamma_1\\ b & d & e & \gamma_2\\ c & e & f & \gamma_3\\ a_1 & a_2 & a_3 & 0\end{vmatrix} & \begin{vmatrix} a & b & c & \gamma_1\\ b & d & e & \gamma_2\\ c & e & f & \gamma_3\\ \beta_1 & \beta_2 & \beta_3 & 0\end{vmatrix} & \begin{vmatrix} a & b & c & \gamma_1\\ b & d & e & \gamma_2\\ c & e & f & \gamma_3\\ \gamma_1 & \gamma_2 & \gamma_3 & 0\end{vmatrix}
\end{array}\right\|.$$

35. Denoting $x^m+c_1x^{m-1}+\ldots+c_m$ by a_x, prove that $P(a_1\ldots a_{2n-1})$ is equal to $(-1)^{\frac{1}{2}n(n-1)}\{(n-1)!\}^n$ when $m=n-1$, and vanishes when $m<n-1$.

36. Express in non-determinant notation the circulant whose first row consists of p positive units followed by $n-p$ zeros.

37. If $n!V_n$ be the number of distinct terms in an axi-symmetric determinant of the n^{th} order with zero elements in the principal diagonal, prove that

$$nV_n-(n-1)V_{n-1}-V_{n-2}+\tfrac{1}{2}V_{n-3}=0.$$

38. Prove that

$$C(a,\ br,\ cr^2,\ dr^3,\ er^4)=(-1)^{\frac{1}{2}n(n-1)}P(br^5,\ cr^5,\ dr^5,\ er^5,\ a,\ b,\ c,\ d,\ e),$$

and writing h for r^5 show that $P(bh,\ ch,\ dh,\ eh,\ a,\ b,\ c,\ d,\ e)$ is resolvable into linear factors,—a result including § 149 and § 153 (5).

39. Prove that all the m^{th}-ary minors of an axi-symmetric determinant of the n^{th} order will vanish if $\frac{1}{2}(n-m+1)(n-m+2)$ of them vanish.

40. Prove that the co-factor of $a_1+a_2\ldots+a_{2n-1}$ in $C(a_1,\ a_2,\ \ldots,\ a_{2n-1})$ is expressible as a determinant of the n^{th} order.

41. Prove that, if ω denote one of the imaginary fifth roots of unity, the double circulant

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 & a_4+b_4 & a_5+b_5\\ a_5+b_2 & a_1+b_3 & a_2+b_4 & a_3+b_5 & a_4+b_1\\ a_4+b_3 & a_5+b_4 & a_1+b_5 & a_2+b_1 & a_3+b_2\\ a_3+b_4 & a_4+b_5 & a_5+b_1 & a_1+b_2 & a_2+b_3\\ a_2+b_5 & a_3+b_1 & a_4+b_2 & a_5+b_3 & a_1+b_4\end{vmatrix}$$

has for a factor the quadratic expression

$$(a_1+\omega a_2+\omega^2 a_3+\omega^3 a_4+\omega^4 a_5)\,(a_1+\omega^{-1}a_2+\omega^{-2}a_3+\omega^{-3}a_4+\omega^{-4}a_5)$$
$$-\,(b_1+\omega b_2+\omega^2 b_3+\omega^3 b_4+\omega^4 b_5)\,(b_1+\omega^{-1}b_2+\omega^{-2}b_3+\omega^{-3}b_4+\omega^{-4}b_5)\,.$$

SKEW DETERMINANTS.

§ 155. A determinant in which every element on one side of the principal diagonal is equal in magnitude to its conjugate element but opposite in sign is said to be SKEW with respect to that diagonal. Diagonal elements being *self*-conjugate, determinants such as

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix},$$

which are skew with respect to a zero diagonal, may be looked upon as having the elements of the principal diagonal conditioned in the same way as the other elements.

The word *skew* as here used is meant to be contrasted with *symmetric*, every kind of symmetric determinant being matched by its corresponding skew determinant. But just as the term 'symmetric' is often used in the narrower sense of 'axi-symmetric,' so 'skew' is almost universally taken to mean 'skew with respect to the principal diagonal.' The slightly different sense in which 'skew' has been employed already in connection with circulants should also be noted.

§ 156. *In the case of a zero-axial skew determinant, (1) conjugate lines differ in the signs of their elements and thus only, (2) coaxial minors are themselves zero-axial skew, (3) conjugate minors are equal or differ only in sign, according as they are of even or odd order, (4) the adjugate determinant is skew if of even order and axi-symmetric if of odd order.*

Here, (1) follows from, or is, the definition; (2) follows from the definition; (3) is deduced from (1); and (4) from (3). The first part of (4) is made more definite below (§ 158).

§ 157. *A zero-axial skew determinant of odd order vanishes.*

If the signs of all the elements be changed, the determinant (§ 26) is changed in sign, and yet (§ 24) is unaltered: therefore it is zero.

Consequently its adjugate, above referred to as being axi-symmetric, is (§ 95) also zero.

§ 158. *The adjugate of a zero-axial skew determinant of even order is zero-axial.*

This follows at once from § 152 (2) and § 157.

§ 159. *A zero-axial skew determinant of even order is the second power of a rational function of the elements.*

Let the determinant be

$$\begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} & \cdots\cdots & a_{1,2n-1} & a_{1,2n} \\ -a_{12} & 0 & a_{23} & a_{24} & \cdots\cdots & a_{2,2n-1} & a_{2,2n} \\ -a_{13} & -a_{23} & 0 & a_{34} & \cdots\cdots & a_{3,2n-1} & a_{3,2n} \\ -a_{14} & -a_{24} & -a_{34} & 0 & \cdots\cdots & a_{4,2n-1} & a_{4,2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots \\ -a_{1,2n-1} & -a_{2,2n-} & -a_{3,2n-1} & -a_{4,2n-1} & \cdots\cdots & 0 & a_{2n-1,2n} \\ -a_{1,2n} & -a_{2,2n} & -a_{3,2n} & -a_{4,2n} & \cdots\cdots & -a_{2n-1,2n} & 0 \end{vmatrix} \text{ or } D \text{, say.}$$

Then (§ 91 or § 96 II.)

$$D^{(1,n}_{(1,n} D = D^{(1}_{(1} D^{(n}_{(n} - D^{(1}_{(n} D^{(n}_{(1} .$$

But (§ 157) $\quad D^{(1}_{(1} = 0 = D^{(n}_{(n},$

and (§ 156, 3) $\quad D^{(1}_{(n} = -D^{(n}_{(1} ;$

$$\therefore \quad D = \{D^{(n)}_{(1}\}^2 \div D^{(1,n}_{(1,n}.$$

Hence, if the theorem holds for $D^{(1,n}_{(1,n}$, which is a zero-axial skew determinant of even order $n-2$, it will hold for the next higher case, viz. of D. But it is evident that it holds for a determinant of the second order: therefore it holds generally.

EXAMPLE:—

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = \begin{vmatrix} a & b & c \\ 0 & d & e \\ -d & 0 & f \end{vmatrix}^2 \div \begin{vmatrix} 0 & d \\ -d & 0 \end{vmatrix},$$

$$= (adf - bed + cd^2)^2 \div d^2,$$

$$= (af - be + cd)^2.$$

§ 160. A determinant having the complementary minor of one of its corner elements a zero-axial skew determinant, like $D_{(1}^{(n}$ or the second determinant in the foregoing example, is called a *bordered* zero-axial skew determinant. The word is similarly applied in connection with other special determinant forms.

§ 161. Looking at the coaxial minors

$$\begin{vmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & a_{34} \\ -a_{34} & 0 \end{vmatrix}, \quad \ldots\ldots, \quad \begin{vmatrix} 0 & a_{2n-1,2n} \\ -a_{2n-1,2n} & 0 \end{vmatrix}$$

of D above, we see that their product

$$a_{12}^2 \, a_{34}^2 \, \ldots . \, a_{2n-1,2n}^2$$

is a term of the determinant, and that therefore one square root of D contains the term $+a_{12}a_{34} \ldots a_{2n-1,2n}$ and the other $-a_{12}a_{34} \ldots a_{2n-1,2n}$. That square root of a zero-axial skew determinant of order $2n$ which contains as a positive term the product of the elements in the places (1, 2), (3, 4), …, $(2n-1, 2n)$ is called the PFAFFIAN (function) of the whole of the elements lying on the same side of the zero diagonal as the elements mentioned. Thus $af - be + cd$ is the Pfaffian of

$$\begin{array}{ccc} a & b & c \\ & d & e \\ & & f. \end{array}$$

Assimilating the notation for Pfaffians to that of determinants which in their properties they closely resemble, we may denote the Pfaffian just given by

$$\left|\begin{array}{ccc} a & b & c \\ & d & e \\ & & f \end{array}\right| ;$$

and more shortly

$$\left|\begin{array}{ccc} a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{array}\right| \text{ by } \mathit{ff}(a_2b_3c_4) \text{ or } {}^{|}|a_2b_3c_4| ;$$

and more shortly still

$${}^{|}|a_{12}a_{23}\ldots\ldots a_{2n-1,2n}| \text{ by } \mathit{ff}(a_{1,2n}) \text{ or } {}^{|}|a_{1,2n}| .$$

Thus the fundamental identity of § 159 may be written

$$\left|a_{1,2n}\right|_{a_{rs}=-a_{sr}}^{a_{rr}=0} = {}^{|}|a_{1,2n}|^2 .$$

The earliest notation and that still in common use is umbral; for example,

$$[1, 2, 3, 4] \text{ for } {}^{|}|a_{14}| .$$

A Pfaffian which is of the n^{th} degree in its elements is said to be of the n^{th} order: thus, ${}^{|}|a_{14}|$ is of the 2nd order.

§ 162. The first row of a Pfaffian of the n^{th} order contains $2n-1$ elements: the line through the 1st column and 2nd row contains the same number: so also do the lines through the 2nd column and 3rd row, through the 3rd column and 4th row, and so on to the last column. These $2n$ lines each containing $2n-1$ elements may be called the *frame-lines* of the Pfaffian and numbered 1st, 2nd, 3rd, etc. in order. Evidently every element of the Pfaffian belongs to *two* frame-lines, and is fully specified as to position when the numbers of these are given. In the Pfaffian ${}^{|}|a_{12}a_{23}\ldots\ldots a_{2n-1,2n}|$

or ${}^{|}|a_{1,2n}|$; written more fully in § 163, the suffixes of each element indicate the numbers of the frame-lines to which it belongs, the smaller number being always written first.

The terms *minor* and *adjugate* and the subsidiary terms connected with them are used in regard to Pfaffians just as in regard to determinants. Thus, if the frame-lines of any element be deleted, the Pfaffian whose elements are in order the elements left is called a *first* or *primary* minor of the original Pfaffian and the *complementary* (minor) of the said element. The notation, also, which corresponds to this nomenclature may be made quite analogous for the two functions: for example, the complementary of the element in the place (r, s) of the Pfaffian ff may be denoted by $ff_{(s}^{(r}$, and the adjugate of $ff(a_{1,2n})$ or ${}^{|}|a_{1,2n}|$ by $ff(A_{1,2n})$ or ${}^{|}|A_{1,2n}|$.

§ 163. *A bordered zero-axial skew determinant is expressible as the product of two Pfaffians.*

Returning to § 159 we find it shown that

$$\left\{D_{(1}^{(n}\right\}^2 = D\,D_{(1,n}^{(1,n},$$

so that on extracting the square root we have (§ 161)

$$\begin{vmatrix} a_{12} & a_{13} & a_{14} & \cdots & a_{1,2n} \\ 0 & a_{23} & a_{24} & \cdots & a_{2,2n} \\ -a_{23} & 0 & a_{34} & \cdots & a_{3,2n} \\ -a_{24} & -a_{34} & 0 & \cdots & a_{4,2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{2,2n-1} & -a_{3,2n-1} & -a_{4,2n-1} & \cdots & a_{2n-1,2n} \end{vmatrix} = \left|\begin{matrix} a_{12} & a_{13} & \cdots & a_{1,2n} \\ & a_{23} & \cdots & a_{2,2n} \\ & & \cdots & a_{3,2n} \\ & & & \cdots\cdots \\ & & & a_{2n-1,2n} \end{matrix}\right| \left|\begin{matrix} a_{23} & a_{24} & \cdots & a_{2,2n-1} \\ & a_{34} & \cdots & a_{3,2n-1} \\ & & & \cdots\cdots \\ & & & a_{2n-2,2n-1} \end{matrix}\right|.$$

That the signs of the roots of D and $D_{(1,n}^{(1,n}$ have been taken correctly is evident on noting that $+a_{12}a_{23}a_{34}\ldots a_{2n-1,2n}$ is a term on the one side, and $(+a_{12}a_{34}a_{56}\ldots a_{2n-1,2n})(+a_{23}a_{45}\ldots a_{2n-2,2n-1})$ a term on the other.

In the preceding the determinant is of odd order; but if we put $a_{2n-1,2n} = 1$ and all the other elements of its column equal to 0, the theorem is seen to hold also for even-ordered determinants, the result being

$$\begin{vmatrix} a_{12} & a_{13} & a_{14} & \dots & a_{1,2n-1} \\ 0 & a_{23} & a_{24} & \dots & a_{2,2n-1} \\ -a_{23} & 0 & a_{34} & \dots & a_{3,2n-1} \\ -a_{24} & -a_{34} & 0 & \dots & a_{4,2n-1} \\ \dots & \dots & \dots & \dots & \dots \\ -a_{2,2n-2} & a_{3,2n-2} & a_{4,2n-2} & \dots & a_{2n-2,2n-1} \end{vmatrix} = \left|\begin{matrix} a_{12} & a_{13} \dots a_{1,2n-2} \\ & a_{23} \dots a_{2,2n-2} \\ & \dots\dots\dots \\ & a_{2n-3,2n-2} \end{matrix}\right| \left|\begin{matrix} a_{23} & a_{24} \dots a_{2,2n-1} \\ & a_{34} \dots a_{3,2n-1} \\ & \dots\dots\dots \\ & a_{2n-2,2n-1} \end{matrix}\right| .$$

If the last column in the second identity here given be passed over the others to become the first column, the form will be still more elegant; for the line of zeros in both identities will then divide the determinant into the two Pfaffians on the right.

§ 164. The complementary minor of any element (a_{hk}) outside the principal diagonal of a zero-axial skew determinant is readily seen to be expressible as a bordered zero-axial skew determinant. Hence every such minor is equal to the product of two Pfaffians.

Instead however of viewing this as a deduction from the foregoing theorem it would be more appropriate perhaps to view the said theorem as a particular case of this. For the substitution of h and k for n and 1 at the commencement in no way increases the difficulty of the proof. The two general results are

(1) If $D = \left|a_{1,2n}\right|_{a_{rs}=-a_{sr}}^{a_{rr}=0}$ and $ff = {}^{|}|a_{1,2n}$, then $D_{(k}^{(h} = ff \times ff_{(k}^{(h}$. $(h>k.)$

(2) If $D = \left|a_{1,2n-1}\right|_{a_{rs}=-a_{sr}}^{a_{rr}=0}$ and $ff = {}^{|}|a_{1,2n-1}|$, then $D_{(k}^{(h} = ff^{(h} \times ff^{(k}$.

If the m^{th} row and m^{th} column of the first D be deleted there results a determinant like the second D; hence in regard to the first D we have

$$D_{(m,k}^{(m,h} = ff_{(h}^{(m} \times ff_{(k}^{(m} .$$

§ 165. *Any determinant of the* 2n*th* *order is expressible as a Pfaffian of the* n*th* *order, viz.* $|a_{1,2n}| = \|P_{1,2n}|$, *where* $P_{rs} = (a_{r1}, a_{r2}, \ldots, a_{r,2n})(a_{s2}, -a_{s1}, a_{s4}, -a_{s3}, \ldots, a_{s,2n}, -a_{s,2n-1})$.

Writing $|a_{1;2n}|$ in the form

$$\begin{vmatrix} a_{12} & -a_{11} & a_{14} & -a_{13} & \ldots & a_{1,2n} & -a_{1,2n-1} \\ a_{22} & -a_{21} & a_{24} & -a_{23} & \ldots & a_{2,2n} & -a_{2,2n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{2n,2} & -a_{2n,1} & a_{2n,4} & -a_{2n,3} & \ldots & a_{2n,2n} & -a_{2n,2n-1} \end{vmatrix},$$

and multiplying the two forms together row-wise we have

$$|a_{1,2n}|^2 = |P_{1,2n}|.$$

But it is evident that $P_{rs} = -P_{sr}$ and that $P_{rr} = 0$; hence, on extracting the square root, the theorem follows.

Treating the rows of $|a_{1,2n}|$ in the same way as the columns have been treated, and multiplying the result column-wise by $|a_{1,2n}|$, we obtain a different but equivalent Pfaffian.

§ 166. *Any determinant is expressible as a Pfaffian of the same order.*

Turning to the identity established in the first part of § 137, and putting the element in the place (rs) of the one factor equal to a_{rs}, and the element in the place (sr) of the other factor equal to $-a_{sr}$, we have the two factors equal in magnitude, and the identity becomes

$$(-1)^m|a_{1m}|^2 = \begin{vmatrix} \frac{1}{2}(a_{11}-a_{11}) & \frac{1}{2}(a_{12}-a_{21}) & \ldots & \frac{1}{2}(a_{12}+a_{21}) & \frac{1}{2}(a_{11}+a_{11}) \\ \frac{1}{2}(a_{21}-a_{12}) & \frac{1}{2}(a_{22}-a_{22}) & \ldots & \frac{1}{2}(a_{22}+a_{22}) & \frac{1}{2}(a_{21}+a_{12}) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{2}(a_{21}+a_{12}) & \frac{1}{2}(a_{22}+a_{22}) & \ldots & \frac{1}{2}(a_{22}-a_{22}) & \frac{1}{2}(a_{21}-a_{12}) \\ \frac{1}{2}(a_{11}+a_{11}) & \frac{1}{2}(a_{12}+a_{21}) & \ldots & \frac{1}{2}(a_{12}-a_{21}) & \frac{1}{2}(a_{11}-a_{11}) \end{vmatrix}.$$

But this centro-symmetric determinant is zero-axial and becomes skew with respect to its zero-diagonal if the signs of all the elements in the last m rows be altered: hence, on extracting the square root, the theorem is established.

It should be noted that the Pfaffian obtained is axi-symmetric.

§ 167. *A Pfaffian of the* n*th order may be expressed as the aggregate of* 2n−1 *products, obtained by multiplying each element of the first row by its complementary and taking the signs of the products alternately positive and negative; in symbols,*

$$\mathrm{ff}(a_{1,2n}) = a_{12}\mathrm{ff}^{(1}_{(2} - a_{13}\mathrm{ff}^{(1}_{(3} + a_{14}\mathrm{ff}^{(1}_{(4} - \ldots + a_{1n}\mathrm{ff}^{(1}_{(n}.$$

The second power of the left-hand member is D, the first determinant written in § 163. Raising the right-hand member to the second power, and remembering (§§ 159, 164) that

$$\left\{\mathit{ff}^{(1}_{(r}\right\}^2 = D^{(r,1}_{(1,r} \quad \text{and} \quad \left\{\mathit{ff}^{(1}_{(r}\right\}\left\{\mathit{ff}^{(1}_{(s}\right\} = D^{(1,r}_{(1,s}.$$

we have

$$\begin{aligned}
&a_{12}^2 D^{(1,2}_{(1,2} - 2a_{12}a_{13}D^{(1,2}_{(1,3} + 2a_{12}a_{14}D^{(1,2}_{(1,4} - \ldots. + 2a_{12}a_{1n}D^{(1,2}_{(1,n}\\
&+ a_{13}^2 D^{(1,3}_{(1,3} - 2a_{13}a_{14}D^{(1,3}_{(1,4} + \ldots\ldots\ldots\ldots - 2a_{13}a_{1n}D^{(1,3}_{(1,n}\\
&+ \ldots\ldots\ldots\\
&+ a_{1n}^2 D^{(1,n}_{(1,n},
\end{aligned}$$

which (§ 62; cf. § 141) is also equal to D. Hence the two expressions $a_{12}\mathit{ff}^{(1}_{(2} - a_{13}\mathit{ff}^{(1}_{(3} + \ldots$ and $\mathit{ff}(a_{1,2n})$ differ at most only in sign. But the first will give the term $a_{12}a_{34}\ldots a_{2n-1,2n}$ positive, and in the second this term is positive by definition: hence the identity is established.

The theorem may also be written in the forms

$${}'|a_{1,2n}| = a_{12}A_{12} + a_{13}A_{13} + a_{14}A_{14} + \ldots + a_{1n}A_{1n};$$

$$\mathit{ff}(a_{1,2n}) = a_{12}\frac{\partial \mathit{ff}}{\partial a_{12}} + a_{13}\frac{\partial \mathit{ff}}{\partial a_{13}} + a_{14}\frac{\partial \mathit{ff}}{\partial a_{14}} + \ldots + a_{1n}\frac{\partial \mathit{ff}}{\partial a_{1n}}.$$

§ 168. In the preceding we have a first instance of a property of Pfaffians perfectly analogous to a property of determinants. The analogy thus exemplified extends with varying degrees of closeness almost throughout the range of both subjects. Thus, in the theorems corresponding to §§ 23, 34 there is a slight difference of statement, viz. we have

> (*1*) *Of the full number of terms of a Pfaffian there is one more positive than negative,*

and

> (*2*) *If two adjacent frame-lines of a Pfaffian be transposed and their common element be changed in sign, the new Pfaffian differs only in sign from the original;*

in the theorems corresponding to those of §§ 35, 36 the divergence increases; but on the other hand the important theorems of §§ 95, 96, 98 may be transferred into the theory of Pfaffians without the slightest alteration.

The student will find it most instructive and interesting to return to § 16, and, framing a definition of a Pfaffian similar to that there given of a determinant, to take §§ 17, 18, ... in succession and try to discover the Pfaffian analogue of every statement that is made.

§ 169. A skew determinant which is not zero-axial may be expressed (§ 63) in terms of the diagonal elements and skew determinants which are zero-axial. Thus

$$\begin{vmatrix} x_1 & a_{12} & a_{13} & a_{14} \\ -a_{12} & x_2 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & x_3 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & x_4 \end{vmatrix} = \left| \begin{array}{ccc} a_{12} & a_{13} & a_{14} \\ & a_{23} & a_{24} \\ & & a_{34} \end{array} \right|^2 \begin{array}{l} + x_1x_2a_{34}^2 + x_1x_3a_{24}^2 \\ + x_1x_4a_{23}^2 + x_2x_3a_{14}^2 \\ + x_2x_4a_{13}^2 + x_3x_4a_{12}^2 \\ + x_1x_2x_3x_4, \end{array}$$

those terms vanishing (§ 157) which have an odd number of diagonal elements; and

$$\begin{vmatrix} x_1 & a_{12} & a_{13} & a_{14} & a_{15} \\ -a_{12} & x_2 & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & x_3 & a_{34} & a_{35} \\ -a_{14} & -a_{24} & -a_{34} & x_4 & a_{45} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & x_5 \end{vmatrix} = \begin{array}{l} x_1\,\|a_{23}a_{24}a_{45}|^2 + x_2\,\|a_{13}a_{34}a_{45}|^2 + \dots\dots \\ + x_1x_2x_3a_{45}^2 + x_1x_2x_4a_{35}^2 + \dots\dots \\ + x_1x_2x_3x_4x_5, \end{array}$$

those terms now vanishing (§ 157) which have an even number of diagonal elements.

When $x_1 = x_2 = \dots = x$ these become expansions according to ascending even or odd powers of x with all the coefficients sums of squares of Pfaffians.

Example:—From any three quantities l, m, n, form nine others the elements of $|\alpha_1\beta_2\gamma_3|$, so that making

$$\begin{aligned} x &= \alpha_1 X + \alpha_2 Y + \alpha_3 Z, \\ y &= \beta_1 X + \beta_2 Y + \beta_3 Z, \\ z &= \gamma_1 X + \gamma_2 Y + \gamma_3 Z, \end{aligned}$$

we may have $x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2$.

Whatever ξ, η, ζ may be, if we put

$$\left.\begin{aligned} x &= \xi + l\eta + m\zeta \\ y &= -l\xi + \eta + n\zeta \\ z &= -m\xi - n\eta + \zeta \end{aligned}\right\} \dots\dots\dots\dots(\text{I.})$$

and

$$\left.\begin{aligned} X &= \xi - l\eta - m\zeta \\ Y &= l\xi + \eta - n\zeta \\ Z &= m\xi + n\eta + \zeta \end{aligned}\right\} \dots\dots\dots\dots(\text{II.})$$

we ensure that $x^2 + y^2 + z^2$ shall be equal to $X^2 + Y^2 + Z^2$: and it therefore only remains to determine x, y, z in terms of X, Y, Z. This may of course be done by solving for ξ, η, ζ in (II.) and substituting in (I.): but the following method is neater. Denoting, for shortness' sake, the skew determinant

$$\begin{vmatrix} 1 & l & m \\ -l & 1 & n \\ -m & -n & 1 \end{vmatrix} \text{ by } \Delta,$$

we have (p. 73) from (II.)

$$\left.\begin{array}{rl} \xi\Delta = & X\Delta^{(1}_{(1} - Y\Delta^{(2}_{(1} + Z\Delta^{(3}_{(1} \\ \eta\Delta = & -X\Delta^{(1}_{(2} + Y\Delta^{(2}_{(2} - Z\Delta^{(3}_{(2} \\ \zeta\Delta = & X\Delta^{(1}_{(3} - Y\Delta^{(2}_{(3} + Z\Delta^{(3}_{(3} \end{array}\right\}\ldots\ldots(a.)$$

But from (I.) and (II.) by addition there results

$$x + X = 2\xi, \qquad y + Y = 2\eta, \qquad z + Z = 2\zeta,$$

so that on substituting these values of 2ξ, 2η, 2ζ in (a) we have

$$\left.\begin{array}{rl} x = & \left\{2\Delta^{(1}_{(1} - \Delta\right\}X - 2\Delta^{(2}_{(1}Y + 2\Delta^{(3}_{(1}Z \\ y = & -2\Delta^{(1}_{(2}X + \left\{2\Delta^{(2}_{(2} - \Delta\right\}Y - 2\Delta^{(3}_{(2}Z \\ z = & 2\Delta^{(1}_{(3}X - 2\Delta^{(2}_{(3}Y + \left\{2\Delta^{(3}_{(3} - \Delta\right\}Z \end{array}\right\}.$$

Hence the required values of $\alpha_1, \alpha_2, \alpha_3, \ldots.$ are

$$2\Delta^{(1}_{(1} - \Delta, \quad -2\Delta^{(2}_{(1}, \quad 2\Delta^{(3}_{(1}, \ldots\ldots$$

EXERCISES. SET XVII.

1. Show that a continuant is expressible as a skew determinant.

2. Verify the identity

$$\left|\begin{array}{ccc} i & l & n \\ & \left|\begin{array}{ccc} b & c & e \\ & g & i \\ & & l \end{array}\right| & \left|\begin{array}{ccc} b & d & e \\ & h & i \\ & & n \end{array}\right| \\ & & \left|\begin{array}{ccc} c & d & e \\ & k & l \\ & & n \end{array}\right| \end{array}\right| = e\left|\begin{array}{ccc} i & l & n \\ & g & h \\ & & k \end{array}\right|.$$

3. Find the number of terms in a Pfaffian of the n^{th} order.

4. Find the product of

$$\left|\begin{array}{rrrr} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{array}\right| \quad \text{and} \quad \left|\begin{array}{rrrr} \alpha & \beta & \gamma & \delta \\ -\beta & \alpha & -\delta & \gamma \\ -\gamma & \delta & \alpha & -\beta \\ -\delta & -\gamma & \beta & -\alpha \end{array}\right|,$$

and thence show how the product of two sums of four squares may be itself expressed as a sum of four squares (cf. Ex. 11, Set XII.).

5. Prove that

$$\left|\begin{array}{ccc} a & b & c \\ & d & e \\ & & f \end{array}\right|(\mu_1+\mu_2+\mu_3+\mu_4) = \left|\begin{array}{ccc} a\mu_2 & b\mu_3 & c\mu_4 \\ & d & e \\ & & f \end{array}\right| + \left|\begin{array}{ccc} a\mu_1 & b & c \\ & d\mu_3 & e\mu_4 \\ & & f \end{array}\right| + \left|\begin{array}{ccc} a & b\mu_1 & c \\ & d\mu_2 & e \\ & & f\mu_4 \end{array}\right| + \left|\begin{array}{ccc} a & b & c\mu_1 \\ & d & e\mu_2 \\ & & f\mu_3 \end{array}\right|.$$

6. Show that the diagonal or axial term of a Pfaffian is always positive.

7. Show that the sign of $a_{23}a_{45}a_{67}\dots a_{2n-2,2n-1}a_{1,2n}$ in $\mathit{ff}(a_{1,2n})$ is +.

8. Find the differential coefficient of a zero-axial skew determinant with respect to one of its non-zero elements.

9. Express as the sum of four Pfaffians with monomial elements the Pfaffian

$$\left|\,\begin{array}{ccc} a_1+a_7 & a_3+a_6 & a_4+a_5 \\ & b_3+b_6 & b_4+b_5 \\ & & c_4+c_5 \end{array}\right|.$$

10. Prove that the last frame-line of a Pfaffian may be passed over the others to occupy the first place without altering the value of the Pfaffian.

11. Show that

$$\left|\,\begin{array}{ccc} |a_1b_2| & |a_1b_3| & |a_1b_4| \\ & |a_2b_3| & |a_2b_4| \\ & & |a_3b_4| \end{array}\right| = 0.$$

12. Show that the co-factor of a_{rs} in $\mathit{ff}(a_{1,2n})$ is $(-1)^{r+s-1}\mathit{ff}^{(r}_{(s}$.

13. Prove that if D be an even-ordered zero-axial skew determinant whose corresponding Pfaffian is ff, then

$$D^{(r}_{(s} : D^{(h}_{(k} :: \mathit{ff}^{(r}_{(s} : \mathit{ff}^{(h}_{(k}.$$

14. Show that

$$\left|\,\begin{array}{ccccc} a & b & c & d & e \\ & f & g & h & i \\ & & \omega j & \omega k & \omega l \\ & & & \omega m & \omega n \\ & & & & \omega p \end{array}\right| - \omega \left|\,\begin{array}{ccccc} \omega a & b & c & d & e \\ & f & g & h & i \\ & & j & k & l \\ & & & m & n \\ & & & & p \end{array}\right|.$$

What if the elements of the second row of the first Pfaffian were also multiplied by ω?

15. Prove that

$$-3\zeta^{\frac{1}{2}}(x_1x_2x_3x_4) = \left|\,\begin{array}{ccc} (x^4-x_3)^3 & (x_4-x_2)^3 & (x_4-x_1)^3 \\ & (x_3-x_2)^3 & (x_3-x_1)^3 \\ & & (x_2-x_1)^3 \end{array}\right|,$$

and give the general theorem.

16. Express a Pfaffian with a frame-line of binomial elements as the sum of two Pfaffians with monomial elements, and squaring both sides deduce the theorem of § 163.

17. Show that an axi-symmetric Pfaffian is expressible as a determinant.

18. Show that

$$\begin{vmatrix} a & b & c \\ & d & e \\ & & f \end{vmatrix} \times \begin{vmatrix} \alpha & \beta & \gamma \\ & \delta & \varepsilon \\ & & \zeta \end{vmatrix} = \begin{vmatrix} d & e & -a\alpha & a\delta & a\varepsilon \\ & f & b\alpha & b\delta & b\varepsilon \\ & & -c\alpha & c\delta & c\varepsilon \\ & & & \beta & \gamma \\ & & & & \zeta \end{vmatrix}.$$

19. Show that a zero-axial skew Pfaffian of the $(2n-1)^{\text{th}}$ order vanishes, and that one of the $2n^{\text{th}}$ order is expressible as the difference of two squares of sums of Pfaffians of the n^{th} order.

20. Express the skew determinant

$$\begin{vmatrix} x & a & b & c \\ -a & -n_1n_2x & n_1c & n_2b \\ -b & -n_1c & n_1n_3x & n_3a \\ -c & -n_2b & -n_3a & -n_2n_3x \end{vmatrix}$$

as the second power of a rational function of the elements.

Establish the identities—

21. $$\begin{vmatrix} 0 & a\mu_1\mu_2 & b\mu_3 & c\mu_4 \\ -a & 0 & d & e \\ -b & -d\mu_3 & 0 & f \\ -c & -e\mu_4 & -f & 0 \end{vmatrix} + \begin{vmatrix} 0 & a\mu_2 & b\mu_1\mu_3 & c\mu_4 \\ -a & 0 & d\mu_2 & e \\ -b & -d & 0 & f \\ -c & -e & -f\mu_4 & 0 \end{vmatrix} + \begin{vmatrix} 0 & a\mu_2 & b\mu_3 & c\mu_1\mu_4 \\ -a & 0 & d & e\mu_2 \\ -b & -d & 0 & f\mu_3 \\ -c & -e & -f & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a\mu_2 & b\mu_3 & c\mu_4 \\ & d & e \\ & & f \end{vmatrix} \left\{ \begin{vmatrix} a\mu_1 & b & c \\ & d\mu_3 & e\mu_4 \\ & & f \end{vmatrix} + \begin{vmatrix} a & b\mu_1 & c \\ & d\mu_2 & e \\ & & f\mu_4 \end{vmatrix} + \begin{vmatrix} a & b & c\mu_1 \\ & d & e\mu_2 \\ & & f\mu_3 \end{vmatrix} \right\}.$$

22. $$\begin{vmatrix} 0 & a & b\mu_1 & c \\ -a\mu_1 & 0 & d\mu_2\mu_3 & e\mu_4 \\ -b & -d & 0 & f \\ -c & -e & -f\mu_4 & 0 \end{vmatrix} + \begin{vmatrix} 0 & a & b & c\mu_1 \\ -a\mu_1 & 0 & d\mu_3 & e\mu_2\mu_4 \\ -b & -d & 0 & f\mu_3 \\ -c & -e & -f & 0 \end{vmatrix} + \begin{vmatrix} 0 & a & b & c\mu_1 \\ -a & 0 & d & e\mu_2 \\ -b\mu_1 & -d\mu_2 & 0 & f\mu_3\mu_4 \\ -c & -e & -f & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a\mu_1 & b & c \\ & d\mu_3 & e\mu_4 \\ & & f \end{vmatrix} \left\{ \begin{vmatrix} a & b\mu_1 & c \\ & d\mu_2 & e \\ & & f\mu_4 \end{vmatrix} + \begin{vmatrix} a & b & c\mu_1 \\ & d & e\mu_2 \\ & & f\mu_3 \end{vmatrix} \right\} + \begin{vmatrix} a & b\mu_1 & c \\ & d\mu_2 & e \\ & & f\mu_4 \end{vmatrix} \begin{vmatrix} a & b & c\mu_1 \\ & d & e\mu_2 \\ & & f\mu_3 \end{vmatrix}.$$

23. Prove that a centre-skew determinant of odd order is equal to the centre element multiplied by its co-factor in the determinant, and that the said co-factor is a centro-symmetric determinant of even order.

24. Prove that a symmetric Pfaffian of the $2n^{\text{th}}$ order is expressible as a Pfaffian of the n^{th} order.

Establish the identities—

25.
$$e^2a\begin{vmatrix} a & b & c & d & e \\ -b & f & g & h & i \\ -c & -g & j & k & l \\ -d & -h & -k & m & n \\ -e & -i & -l & -n & 0 \end{vmatrix} = \begin{vmatrix} ea & ia & la & na \\ -ia & ef & \begin{vmatrix} b & c & e \\ & g & i \\ & & l \end{vmatrix} & \begin{vmatrix} b & d & e \\ & h & i \\ & & n \end{vmatrix} \\ -la & -\begin{vmatrix} b & c & e \\ & g & i \\ & & l \end{vmatrix} & ej & \begin{vmatrix} c & d & e \\ & k & l \\ & & n \end{vmatrix} \\ -na & -\begin{vmatrix} b & d & e \\ & h & i \\ & & n \end{vmatrix} & -\begin{vmatrix} c & d & e \\ & k & l \\ & & n \end{vmatrix} & em \end{vmatrix}.$$

26.
$$\begin{vmatrix} a & b & c & d & e \\ -b & f & g & h & i \\ -c & -g & j & k & l \\ -d & -h & -k & 0 & n \\ -e & -i & -l & -n & 0 \end{vmatrix} = \begin{vmatrix} na & \begin{vmatrix} b & d & e \\ & h & i \\ & & n \end{vmatrix} & \begin{vmatrix} c & d & e \\ & k & l \\ & & n \end{vmatrix} \\ \begin{vmatrix} b & d & e \\ & h & i \\ & & n \end{vmatrix} & nf & \begin{vmatrix} g & h & i \\ & k & l \\ & & n \end{vmatrix} \\ -\begin{vmatrix} c & d & e \\ & k & l \\ & & n \end{vmatrix} & -\begin{vmatrix} g & h & i \\ & k & l \\ & & n \end{vmatrix} & nj \end{vmatrix}.$$

27. Prove that the product of two symmetric Pfaffians is expressible as a symmetric Pfaffian.

28. If $1.3.5\dots(2n-1)V_n$ be the number of distinct terms in a zero-axial skew determinant of the $2n^{\text{th}}$ order, prove that

$$V_n = (2n-1)V_{n-1} - (n-1)V_{n-2},$$

and calculate $V_2, V_3, \dots, V_6$. Show also that

$$2^n V_n = 1 + 1.n + 1.5C_{n,2} + 1.5.9C_{n,3} + \dots$$

29. Prove that any zero-axial skew Pfaffian of the $2n^{\text{th}}$ order is expressible as the product of two Pfaffians of the n^{th} order: and, conversely, that the product of any two Pfaffians of the n^{th} order is expressible as a zero-axial skew Pfaffian of the $2n^{\text{th}}$ order.

COMPOUND DETERMINANTS.

§ 170. A determinant with elements which are themselves determinants is called a COMPOUND determinant.

Several important theorems on compound determinants have already been given, viz. the theorem (§ 95) regarding the determinant of the primary minors (or adjugate) of a determinant, the theorem (§ 96) regarding a minor of the said adjugate, and the theorems of §§ 93, 99. The first two of these at once suggest a more general subject for consideration, viz. the determinant of the m^{th}-ary minors.

§ 171. In the case of a determinant of the n^{th} order it is possible to delete m rows in $C_{n,m}$ different ways, and to delete m columns in $C_{n,m}$ different ways; and thus the number of minors of the $(n-m)^{\text{th}}$ order belonging to a determinant of the n^{th} order is $(C_{n,m})^2$. Arranging these minors as the elements of a new determinant, and giving precedence in any row to that minor of two which first has a column number less than the corresponding column number of the other, and precedence in any column to that minor of two which first has a row number less than the corresponding row number of the other, we obtain what is known as *the determinant of the* m^{th}*-ary minors* of the original determinant, or, more conveniently and shortly, *the* $(n-m)^{\text{th}}$ *Compound* of the original determinant. Just as the $(n-1)^{\text{th}}$ or highest compound of a determinant is called the adjugate of the determinant, so, generally, the $(n-m)^{\text{th}}$ compound is called the adjugate of the m^{th} compound.

EXAMPLE:—The determinant of the secondary minors—the third compound —of

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}$$

is

$$\left|\begin{array}{ccccc} |a_1b_2c_3| & |a_1b_2c_4| & |a_1b_2c_5| & \ldots\ldots & |a_3b_4c_5| \\ |a_1b_2d_3| & |a_1b_2d_4| & |a_1b_2d_5| & \ldots\ldots & |a_3b_4d_5| \\ |a_1b_2e_3| & |a_1b_2e_4| & |a_1b_2e_5| & \ldots\ldots & |a_3b_4e_5| \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ |c_1d_2e_3| & |c_1d_2e_4| & |c_1d_2e_5| & \ldots\ldots & |c_3d_4e_5| \end{array}\right|,$$

or

$$\left|\,|a_1b_2c_3| \quad |a_1b_2d_4| \quad |a_1b_2e_5| \quad \ldots\ldots \quad |c_3d_4e_5|\,\right|.$$

In the umbral notation (§ 102), which makes more evident the mode of arranging the minors, it is

$$\left|\begin{array}{ccccc} \begin{vmatrix}1&2&3\\1&2&3\end{vmatrix} & \begin{vmatrix}1&2&3\\1&2&4\end{vmatrix} & \begin{vmatrix}1&2&3\\1&2&5\end{vmatrix} & \cdots & \begin{vmatrix}1&2&3\\3&4&5\end{vmatrix} \\ \begin{vmatrix}1&2&4\\1&2&3\end{vmatrix} & \begin{vmatrix}1&2&4\\1&2&4\end{vmatrix} & \begin{vmatrix}1&2&4\\1&2&5\end{vmatrix} & \cdots & \begin{vmatrix}1&2&4\\3&4&5\end{vmatrix} \\ \begin{vmatrix}1&2&5\\1&2&3\end{vmatrix} & \begin{vmatrix}1&2&5\\1&2&4\end{vmatrix} & \begin{vmatrix}1&2&5\\1&2&5\end{vmatrix} & \cdots & \begin{vmatrix}1&2&5\\3&4&5\end{vmatrix} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \begin{vmatrix}3&4&5\\1&2&3\end{vmatrix} & \begin{vmatrix}3&4&5\\1&2&4\end{vmatrix} & \begin{vmatrix}3&4&5\\1&2&5\end{vmatrix} & \cdots & \begin{vmatrix}3&4&5\\3&4&5\end{vmatrix} \end{array}\right|,$$

or

$$\left|\,\begin{vmatrix}1&2&3\\1&2&3\end{vmatrix}, \quad \begin{vmatrix}1&2&4\\1&2&4\end{vmatrix}, \quad \begin{vmatrix}1&2&5\\1&2&5\end{vmatrix}, \ldots, \quad \begin{vmatrix}3&4&5\\3&4&5\end{vmatrix}\,\right|;$$

where in any row or column the order of precedence is decided by the order of magnitude of the numbers 123, 124, 125, 134, 135, 145, 234, 235, 245, 345, whose digits are the column or row numbers of the minors.

§ 172. *The* m*th* *compound of* $|a_{1n}|$ *is of the same order as the* (n−m)*th* *compound, (2) conjugate elements in it are conjugate minors of* $|a_{1n}|$, *(3) if each element of it be replaced by the complementary minor in* $|a_{1n}|$ *the result is equal to the* (n−m)*th* *compound.*

Here (1) depends on the fact that $C_{n,m} = C_{n,n-m}$, and (3) upon § 39.

§ 173. *The* m*th* *compound of a minor of* $|a_{1n}|$ *is a minor of the* m*th* *compound of* $|a_{1n}|$.

§ 174. *The* m*th* *compound of* $|a_{1n}|$ *is equal to* $|a_{1n}|^{C_{n-1,m-1}}$.

Let Δ_m denote the m^{th} compound of Δ, so that Δ_1 and Δ are the same. If in Δ_m we prefix to each element the sign + or – according as the sum of its row and column numbers is even or odd, the signs in any row will be in order all the same as those in any other row, or all opposite, and the signs of conjugate elements will (§ 172, 2) be alike; the change made is thus equivalent to altering the signs of all the elements in certain rows and afterwards all the elements in the corresponding columns, hence, the value of the determinant itself will remain unaltered. Multiplying Δ_m as thus changed by Δ_{n-m} in the form of § 172, 3 we have a determinant whose principal diagonal elements are (§ 77) all Δ, and other elements (§ 78) all 0: hence

$$\Delta_m \Delta_{n-m} = (\Delta)^{C_{n,m}}$$

But Δ has in general no factors, therefore Δ_m and Δ_{n-m} must both be powers of Δ. Now Δ_m is of the order $C_{n,m}$ and each element of it is of the m^{th} degree in the elements of Δ: consequently the power which Δ_m is of Δ has for its index $mC_{n,m} \div n$, that is, $C_{n-1,m-1}$.

§ 175. *If the* m^{th} *compound of* $|\mathrm{a}_{1n}|$ *be formed, any minor of it of the* k^{th} *order is equal to the product obtained by multiplying the complementary of the corresponding minor in the adjugate compound by* $|\mathrm{a}_{1n}|^{\mathrm{k}-\mathrm{C}_{n-1,m}}$.

This is related to the theorem which precedes it exactly as the theorem of § 96 is related to that of § 95. The proof of § 96 with evident modifications need not therefore be repeated.

§ 176. For certain of the minors (see § 173) of the m^{th} compound an expression of quite different form from the foregoing may be obtained by means of § 174: new identities thus arise.

Another special minor is that dealt with in § 93, viz.

$$\left\|\begin{vmatrix}1,2,..,m-1,m\\1,2,..,m-1,m\end{vmatrix}, \begin{vmatrix}1,2,..,m-1,m+1\\2,3,.., m, m+1\end{vmatrix}, \begin{vmatrix}1,2,..,m-1,m+2\\3,4,..,m+1,m+2\end{vmatrix},...., \begin{vmatrix}1,2,...., m-1,n\\n-m+1,.....,n\end{vmatrix}\right\|,$$

which is there shown to be equal to

$$\begin{vmatrix}1,2,...,n\\1,2,...,n\end{vmatrix} \times \begin{vmatrix}1,2,...,m-1\\2,3,..., m\end{vmatrix} \times \begin{vmatrix}1,2,...,m-1\\3,4,...,m+1\end{vmatrix} \times ...... \times \begin{vmatrix}1,2,........,m-1\\n-m+1,..., n-1\end{vmatrix}.$$

Still another is that with which the Complementary of this theorem is concerned (§ 99); and there are many more having the like property, viz. expressibility as a product of powers of the original determinant and a number of its minors.

§ 177. The compound determinants considered up to this point all belong to one class, viz. that in which the group of row numbers in the first element of any row is repeated throughout the row, and the group of column numbers in the first element of any column is repeated throughout the column; and in which, consequently, the determinant is specified when a diagonal is given. Evidently there are three other classes which might be proposed for consideration, viz. (2) that in which the groups of row numbers vary neither in rows nor columns and the groups of column numbers vary in both, (3) that in which the groups of row numbers vary both in rows and columns and the groups of column numbers vary only in rows, (4) that in which the groups of row numbers vary both in rows and columns and the groups of column numbers do so likewise.

Compound determinants of the third and fourth classes have not as yet been investigated. The following is an important theorem regarding a special determinant of the second class.

§ 178. *If deleting the first set of* r *columns of* $|a_{n+1,2n}|$ *and replacing the set in succession by the* $C_{n,r}$ *sets of* r *columns of* $|a_{1n}|$ *we take the determinants thus obtained as the elements of the first column of a new determinant* **D**, *and deleting the second, third, ..., $(C_{n,r})^{th}$ sets of* r *columns of* $|a_{n+1,2n}|$ *we obtain in like manner the remaining columns of* **D**; *then* $\mathbf{D} = |a_{1n}|^{C_{n-1,r-1}} \times |a_{n+1,2n}|^{C_{n-1,r}}$.

Expressing each element of **D** as a sum of products of complementary minors, one factor of the product being formed from the $n-r$ columns which belong to $|a_{n+1,2n}|$ and the other from the r columns which belong to $|a_{1,n}|$, we see that **D** may be obtained by taking the $(n-r)^{\text{th}}$ compound of $|a_{n+1,2n}|$ with its rows in reversed order and the r^{th} compound of $|a_{1,n}|$ with its columns in reversed order, and multiplying the two compounds together column-wise. Hence (§ 174)

$$\begin{aligned} \mathbf{D} &= |a_{n+1,2n}|^{C_{n-1,n-r-1}} \times |a_{1n}|^{C_{n-1,r-1}} \\ &= |a_{n+1,2n}|^{C_{n-1,r}} \times |a_{1n}|^{C_{n-1,r-1}}. \end{aligned}$$

If each of the principal diagonal elements of $|a_{n+1,2n}|$ be made unity and all the other elements zero, this theorem degenerates into that of § 174.

§ 179. *If any identical relation be established between a number of the minors of a determinant or between the determinant itself and a number of its minors, the elements of the determinant being letters with single suffixes and the determinants denoted by means of their principal diagonals, then a new theorem is always obtainable by merely taking a line of new letters with new suffixes and annexing it to the end of the diagonal of every determinant, including those of order 0, occurring in the identity.*

Let (**A**) be the established identity, and $|a_1 b_2 c_3 \ldots l_n|$ the determinant whose minors are involved in it. Taking the Complementary (§ 98) of (**A**) with respect to $|a_1b_2c_3\ldots l_n|$ we obtain an identity, (**B**) say, likewise involving minors of $|a_1b_2c_3\ldots l_n|$. But these minors are also minors of $|a_1b_2c_3\ldots l_n r_\alpha s_\beta \ldots z_\omega|$, and therefore it is allowable to take the Complementary of (**B**) with respect to this extended determinant. Doing this we pass, not back to (**A**), but to a new theorem (**A**′) which is seen to be derivable from (**A**) by annexing to the end of the diagonal of every determinant in it the line of letters $r_\alpha s_\beta \ldots z_\omega$. The clause "including those of order 0" is necessitated by the last clause in the enunciation of the Law of Complementaries.

This theorem is the *Law of Extensible Minors* incidentally exemplified by (*B*) p. 143. It might indeed have been enunciated there, as its application is not limited to theorems regarding the present portion of our subject.

§ 180. By means of the Law of Extensible Minors every identity which we have given in Compound Determinants may be made more general. Thus, taking the identity (§ 176)

$$\begin{Vmatrix} |a_1b_2| & |a_2b_3| & |a_3b_4| \\ |a_1c_2| & |a_2c_3| & |a_3c_4| \\ |a_1d_2| & |a_2d_3| & |a_3d_4| \end{Vmatrix} = |a_1b_2c_3d_4|\,a_2a_3, \qquad (a)$$

and adopting the extension e_5f_6, we have

$$\begin{Vmatrix} |a_1b_2e_5f_6| & |a_2b_3e_5f_6| & |a_3b_4e_5f_6| \\ |a_1c_2e_5f_6| & |a_2c_3e_5f_6| & |a_3c_4e_5f_6| \\ |a_1d_2e_5f_6| & |a_2d_3e_5f_6| & |a_3d_4e_5f_6| \end{Vmatrix} = |a_1b_2c_3d_4e_5f_6|\cdot|a_2e_5f_6|\cdot|a_3e_5f_6|;$$

or, if we view the elements of $|a_1b_2c_3d_4|$ in (a) as themselves determinants of order 1, we have

$$|e_5f_6|^3\cdot\begin{Vmatrix} |a_1b_2e_5f_6| & |a_2b_3e_5f_6| & |a_3b_4e_5f_6| \\ |a_1c_2e_5f_6| & |a_2c_3e_5f_6| & |a_3c_4e_5f_6| \\ |a_1d_2e_5f_6| & |a_2d_3e_5f_6| & |a_3d_4e_5f_6| \end{Vmatrix}$$

$$= \begin{Vmatrix} |a_1e_5f_6| & |a_2e_5f_6| & |a_3e_5f_6| & |a_4e_5f_6| \\ |b_1e_5f_6| & |b_2e_5f_6| & |b_3e_5f_6| & |b_4e_5f_6| \\ |c_1e_5f_6| & |c_2e_5f_6| & |c_3e_5f_6| & |c_4e_5f_6| \\ |d_1e_5f_6| & |d_2e_5f_6| & |d_3e_5f_6| & |d_4e_5f_6| \end{Vmatrix} \cdot |a_2e_5f_6| \cdot |a_3e_5f_6|.$$

In corroboration of these new identities we deduce from them

$$\begin{Vmatrix} |a_1e_5f_6| & |a_2e_5f_6| & |a_3e_5f_6| & |a_4e_5f_6| \\ |b_1e_5f_6| & |b_2e_5f_6| & |b_3e_5f_6| & |b_4e_5f_6| \\ |c_1e_5f_6| & |c_2e_5f_6| & |c_3e_5f_6| & |c_4e_5f_6| \\ |d_1e_5f_6| & |d_2e_5f_6| & |d_3e_5f_6| & |d_4e_5f_6| \end{Vmatrix} = |a_1b_2c_3d_4e_5f_6| \cdot |e_5f_6|^3,$$

which is the Extensional of the manifest identity

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = |a_1b_2c_3d_4|,$$

and has been already proved (§ 96).

EXERCISES. SET XVIII.

1. In what row of the second compound of $|a_1b_2c_3d_4|$ does the element $|b_2c_3|$ occur?

2. Find to what row and column the element $|c_1e_5|$ belongs in the determinant of the tertiary minors of $|a_1b_2c_3d_4e_5|$.

3. Find the element in the 1st row and 83rd column of the third compound of $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{vmatrix}$.

4. Establish the identities

$$\left\Vert \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}, \; \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}, \; \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} \right\Vert = 0 = \left\Vert \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}, \; \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}, \; \begin{vmatrix} 3 & 4 \\ 1 & 4 \end{vmatrix} \right\Vert.$$

5. Show that the second compound of a determinant is equal to the ad-

jugate of the latter. Give the general theorem of which this is a case.

6. Show that to every minor of a certain order in the m^{th} compound of a determinant there is an equivalent minor in the adjugate compound.

7. Resolve into factors

$$\left\| \begin{array}{ccc} \begin{vmatrix}1&2&3\\1&4&5\end{vmatrix} & \begin{vmatrix}1&2&3\\1&4&6\end{vmatrix} & \begin{vmatrix}1&2&3\\1&5&6\end{vmatrix} \\ \begin{vmatrix}1&2&3\\2&4&5\end{vmatrix} & \begin{vmatrix}1&2&3\\2&4&6\end{vmatrix} & \begin{vmatrix}1&2&3\\2&5&6\end{vmatrix} \\ \begin{vmatrix}1&2&3\\3&4&5\end{vmatrix} & \begin{vmatrix}1&2&3\\3&4&6\end{vmatrix} & \begin{vmatrix}1&2&3\\3&5&6\end{vmatrix} \end{array} \right\|.$$

8. Show that any minor of $|a_{1n}|$ of the m^{th} order is expressible as the $(C_{m-1,t})^{\text{th}}$ root of a minor of the $(m-t)^{\text{th}}$ compound of $|a_{1n}|$.

9. If the last column of each element of the compound determinant in Ex. 34, p. 194, instead of being taken from $|a_1\beta_2\gamma_3|$ were similarly taken from $|x_1y_2z_3|$, what should the left-hand member become?

10. If M_h be a minor of $|a_{1n}|$ of the h^{th} order, M'_{n-h} being its complementary, and there be formed all the minors of $|a_{1n}|$ of the $(h+k)^{\text{th}}$ order which contain neither all the rows nor all the columns of M_h, show that the determinant whose elements are these minors is equal to

$$(M'_{n-h})^{C_{n-h-1,k}} \times \left|a_{1n}\right|^{C_{n-1,h+k-1}-C_{n-h,k}}.$$

11. Resolve into factors

$$\left\| \begin{array}{ccc} \begin{vmatrix}1&2&3&4\\1&4&5&7\end{vmatrix} & \begin{vmatrix}1&2&3&4\\1&4&6&7\end{vmatrix} & \begin{vmatrix}1&2&3&4\\1&5&6&7\end{vmatrix} \\ \begin{vmatrix}1&2&3&4\\2&4&5&7\end{vmatrix} & \begin{vmatrix}1&2&3&4\\2&4&6&7\end{vmatrix} & \begin{vmatrix}1&2&3&4\\2&5&6&7\end{vmatrix} \\ \begin{vmatrix}1&2&3&4\\3&4&5&7\end{vmatrix} & \begin{vmatrix}1&2&3&4\\3&4&6&7\end{vmatrix} & \begin{vmatrix}1&2&3&4\\3&5&6&7\end{vmatrix} \end{array} \right\|.$$

12. If M_h be a minor of $|a_{1n}|$ of the h^{th} order, M'_{n-h} being its complementary, and there be formed all the minors of $|a_{1n}|$ of the $(n-h-k)^{\text{th}}$ order such that neither all their rows nor all their columns belong to M'_{n-h}, show that the determinant whose elements are those minors is equal to

$$(M_h)^{C_{n-h-1,k}} \times \left|a_{1n}\right|^{C_{n-1,h+k}-C_{n-h-1,k}}.$$

13. Establish the identity

$$\left\| \begin{array}{cccc} \begin{vmatrix}1&2&3\\1&7&8\end{vmatrix} & \begin{vmatrix}1&2&4\\2&7&8\end{vmatrix} & \begin{vmatrix}1&3&4\\3&7&8\end{vmatrix} & \begin{vmatrix}2&3&4\\4&5&6\end{vmatrix} \end{array} \right\| = 0.$$

14. Show that if D′ be the determinant so related to D (§ 178) that any element of it contains those columns of $|a_{1n}|$ and $|a_{n+1,2n}|$ which are not contained in the corresponding element of D, then

$$\mathrm{D}\,\mathrm{D}' = \left\{ |a_{1n}| \cdot |a_{n+1,2n}| \right\}^{C_{n,r}}.$$

Establish the identities

15. $$\left\| \begin{vmatrix} 14 \\ 12 \end{vmatrix}, \begin{vmatrix} 15 \\ 13 \end{vmatrix}, \begin{vmatrix} 24 \\ 14 \end{vmatrix}, \begin{vmatrix} 25 \\ 15 \end{vmatrix}, \begin{vmatrix} 34 \\ 23 \end{vmatrix}, \begin{vmatrix} 35 \\ 24 \end{vmatrix}, \begin{vmatrix} 45 \\ 25 \end{vmatrix} \right\| = - \begin{vmatrix} 12345 \\ 12345 \end{vmatrix}^2 \cdot \begin{vmatrix} 45 \\ 12 \end{vmatrix}^2.$$

16. $$\left\| \begin{vmatrix} 3456 \\ 1234 \end{vmatrix}, \begin{vmatrix} 2456 \\ 1356 \end{vmatrix}, \begin{vmatrix} 2356 \\ 1456 \end{vmatrix}, \begin{vmatrix} 1456 \\ 2346 \end{vmatrix}, \begin{vmatrix} 1356 \\ 2356 \end{vmatrix}, \begin{vmatrix} 1256 \\ 2456 \end{vmatrix} \right\| = - \begin{vmatrix} 123456 \\ 123456 \end{vmatrix}^3 \cdot \begin{vmatrix} 56 \\ 26 \end{vmatrix} \cdot \begin{vmatrix} 56 \\ 34 \end{vmatrix} \cdot \begin{vmatrix} 56 \\ 56 \end{vmatrix}.$$

17. If D_h be a minor of D (Ex. 14) of the hth order, show that the complementary of the corresponding minor in D′ is equal to

$$\mathrm{D}_h \times |a_{1n}|^{C_{n-1,r-h}} \times |a_{n+1,2n}|^{C_{n-1,r-1-h}}.$$

DETERMINANTS WHOSE ELEMENTS ARE DIFFERENTIAL COEFFICIENTS OF A SET OF FUNCTIONS.

§ 181. If there be n functions all of the same n variables, the determinant which in every case has the element in its rth row and sth column equal to the differential coefficient of the rth function with respect to the sth variable is called the JACOBIAN of the set of functions with respect to the said variables. Thus the Jacobian of y_1, y_2, …, y_n with respect to $x_1, x_2, \ldots, x_n$ is

$$\left| \frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_2}, \ldots, \frac{\partial y_n}{\partial x_n} \right| \quad \text{or} \quad \left| \frac{\partial y_1}{\partial x_n} \right|.$$

It is usually denoted by

$$J(y_1, y_2, \ldots, y_n)$$

where there is no possibility of ambiguity in regard to the independent variables.

§ 182. *If* $\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n$ *simultaneously receive the increments* $\Delta_r\mathrm{x}_1, \Delta_r\mathrm{x}_2, \ldots, \Delta_r\mathrm{x}_n$ *respectively, and in consequence of this simultaneous change the functions* $\mathrm{y}_1, \mathrm{y}_2, \ldots, \mathrm{y}_n$ *receive the increments* $\Delta_r\mathrm{y}_1, \Delta_r\mathrm{y}_2, \ldots, \Delta_r\mathrm{y}_n$ *respectively,* *then the limiting value of*

$$|\Delta_1 y_1, \Delta_2 y_2, \ldots, \Delta_n y_n| \div |\Delta_1 x_1, \Delta_2 x_2, \ldots, \Delta_n x_n|$$

when the elements of the latter determinant are indefinitely diminished is equal to

$$\left|\frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_2}, \ldots, \frac{\partial y_n}{\partial x_n}\right|.$$

From § 67 we have

$$\left|\frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_2}, \ldots, \frac{\partial y_n}{\partial x_n}\right| . |\Delta_1 x_1, \Delta_2 x_2, \ldots, \Delta_n x_n| = |Q_{1n}|.$$

where

$$Q_{r,s} = \frac{\partial y_r}{\partial x_1}\Delta_s x_1 + \frac{\partial y_r}{\partial x_2}\Delta_s x_2 + \ldots + \frac{\partial y_r}{\partial x_n}\Delta_s x_n.$$

But in the limiting case referred to, we know that this sum of products is equal to $\Delta_s y_r$, and therefore that $|Q_{1n}|$ is then equal to $|\Delta_1 y_1, \Delta_2 y_2, \ldots, \Delta_n y_n|$. The theorem is thus established.

Either of the two things here shown to be equal, viz.

$$\left|\frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_2}, \ldots, \frac{\partial y_n}{\partial x_n}\right| \text{ and } \mathcal{L}.\frac{|\Delta_1 y_1, \Delta_2 y_2, \ldots, \Delta_n y_n|}{|\Delta_1 x_1, \Delta_2 x_2, \ldots, \Delta_n x_n|},$$

might have been taken as the definition of the *Jacobian* of the set of functions. If the latter instead of the former be adopted, the definition resembles that of the *Differential Coefficient* of a function, the fact at the basis of the resemblance being that the Differential Coefficient is the case of the Jacobian for $n = 1$. The notation

$$\frac{d(y_1, y_2, \ldots, y_n)}{d(x_1, x_2, \ldots, x_n)}$$

for the Jacobian of $y_1, y_2, \ldots, y_n$ with respect to $x_1, x_2, \ldots, x_n$ is valuable as exhibiting this connection.

§ 183. *If* $y_{m+1}, \ldots, y_n$ *be constant with respect to* $x_1 \ldots x_m$ *or* $y_1, \ldots, y_m$ *be constant with respect to* $x_{m+1}, \ldots, x_n$ *then*

$$\frac{d(y_1, \ldots, y_m, y_{m+1}, \ldots, y_n)}{d(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)} = \frac{d(y_1, \ldots, y_m)}{d(x_1, \ldots, x_m)} \times \frac{d(y_{m+1}, \ldots, y_n)}{d(x_{m+}, \ldots, x_n)},$$

and in particular

$$\frac{d(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n)}{d(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)} = \frac{d(y_1, \ldots, y_m)}{d(x_1, \ldots, x_m)}.$$

This follows at once from either of the data and § 79.

§ 184. *If* $y_1, y_2, \ldots, y_n$ *be functions of* $v_1, v_2, \ldots, v_n$, *and* $v_1, v_2, \ldots, v_n$ *be functions of* $x_1, x_2, \ldots, x_n$, *then*

$$\frac{d(y_1, y_2, \ldots, y_n)}{d(x_1, x_2, \ldots, x_n)} = \frac{d(y_1, y_2, \ldots, y_n)}{d(v_1, v_2, \ldots, v_n)} \times \frac{d(v_1, v_2, \ldots, v_n)}{d(x_1, x_2, \ldots, x_n)}.$$

Multiplying together the two Jacobians on the right after changing the columns of the second into rows, we have (§ 67) a determinant $|X_{1n}|$ where

$$X_{rs} = \frac{\partial y_r}{\partial v_1} \cdot \frac{\partial v_1}{\partial x_s} + \frac{\partial y_r}{\partial v_2} \cdot \frac{\partial v_2}{\partial x_s} + \ldots + \frac{\partial y_r}{\partial v_n} \cdot \frac{\partial v_n}{\partial x_s}.$$

But this sum of products we know equals $\frac{\partial y_r}{\partial x_s}$,

$$\therefore \; |X_{1n}| = \left| \frac{\partial y_1}{\partial x_1}, \frac{\partial y_2}{\partial x_2}, \ldots, \frac{\partial x_n}{\partial x_n} \right|,$$

as was to be proved.

§ 185. *If* $y_1, y_2, \ldots y_n$ *be independent functions of* $x_1, x_2, \ldots, x_n$, *the Jacobian of* $y_1, y_2, \ldots, y_n$ *with respect to* $x_1, x_2, \ldots, x_n$ *is the reciprocal of the Jacobian of* $x_1, x_2, \ldots, x_n$ *with respect to* $y_1, y_2, \ldots, y_n$.

Since $y_1, y_2, \ldots, y_n$ are independent functions of $x_1, x_2, \ldots, x_n$ it follows that $x_1, x_2, \ldots, x_n$ are independent functions of $y_1, y_2 \ldots, y_n$: therefore from the preceding theorem the product of the two Jacobians in question

$$= \frac{d(x_1, x_2, \ldots, x_n)}{d(x_1, x_2, \ldots, x_n)}$$

$$= 1.$$

§ 186. *If* $y_1, y_2, \dots, y_n$ *be given implicitly in terms of* $x_1, x_2, \dots, x_n$, *in the form*

$$F_1(x_1, \dots, x_n, y_1, \dots, y_n) = 0,\quad F_2(x_1, \dots, x_n, y_1, \dots, y_n) = 0,\quad \dots\dots,\quad F_n(x_1, \dots, x_n, y_1, \dots, y_n) = 0,$$

then

$$\frac{d(y_1, y_2, \dots, y_n)}{d(x_1, x_2, \dots, x_n)} = (-1)^n \frac{d(F_1, F_2, \dots, F_n)}{d(x_1, x_2, \dots, x_n)} \div \frac{d(F_1, F_2, \dots, F_n)}{d(y_1, y_2, \dots, y_n)}.$$

This, like the identities of §§ 182, 184, is a direct result of the multiplication theorem (§ 67) and a theorem regarding differentiation; the differentiation theorem now being

$$\frac{\partial F_r}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_s} + \frac{\partial F_r}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_s} + \dots + \frac{\partial F_r}{\partial y_n}\cdot\frac{\partial y_n}{\partial x_s} = -\frac{\partial F_r}{\partial x_s}.$$

§ 187. *The Jacobian of a set of* n *functions is always expressible as a product of* n *differential coefficients; viz.*

$$\frac{d(y_1, \dots y_n)}{d(x_1, \dots x_n)} = \frac{\partial \phi_1}{\partial x_1}\cdot\frac{\partial \phi_2}{\partial x_2} \dots \frac{\partial \phi_n}{\partial x_n}$$

where ϕ_r *is a function of* $y_1, \dots, y_{r-1}, x_r, \dots, x_n$.

Since we have y_1 given as a function $x_1, x_2, \dots, x_n$, it follows that x_1 is a function of $y_1, x_2, \dots, x_n$, and therefore also that y_2 is a function of the latter set of variables. Again, from this we have x_2 a function of $y_1, y_2, x_3, \dots, x_n$; therefore also y_3 is a function of $y_1, y_2, x_3, \dots, x_n$. Similarly, it is evident that y_4 is a function of $y_1, y_2, y_3, x_4, \dots, x_n$, and so on; so that we have

$$\begin{aligned}
y_1 - \phi_1(x_1, x_2, \dots, x_n) &= 0\\
y_2 - \phi_2(y_1, x_2, \dots, x_n) &= 0\\
y_3 - \phi_3(y_1, y_2, x_3, \dots, x_n) &= 0\\
\dots\dots\dots\dots\dots\dots\dots\dots&\dots\dots\\
y_n - \phi_n(y_1, y_2, \dots, y_{n-1}, x_n) &= 0.
\end{aligned}$$

Hence from § 186 there results

$$\frac{d(y_1, y_2, \dots, y_n)}{d(x_1, x_2, \dots, x_n)}$$

$$=(-1)^n \begin{vmatrix} -\frac{\partial\phi_1}{\partial x_1} & -\frac{\partial\phi_1}{\partial x_2} & -\frac{\partial\phi_1}{\partial x_3} & \dots & -\frac{\partial\phi_1}{\partial x_n} \\ 0 & -\frac{\partial\phi_2}{\partial x_2} & -\frac{\partial\phi_2}{\partial x_3} & \dots & -\frac{\partial\phi_2}{\partial x_n} \\ 0 & 0 & -\frac{\partial\phi_3}{\partial x_3} & \dots & -\frac{\partial\phi_3}{\partial x_n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\frac{\partial\phi_n}{\partial x_n} \end{vmatrix} \div \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ -\frac{\partial\phi_2}{\partial y_1} & 1 & 0 & \dots & 0 \\ -\frac{\partial\phi_3}{\partial y_1} & -\frac{\partial\phi_3}{\partial y_2} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{\partial\phi_n}{\partial y_1} & -\frac{\partial\phi_n}{\partial y_2} & -\frac{\partial\phi_n}{\partial y_3} & \dots & 1 \end{vmatrix},$$

$$=\frac{\partial\phi_1}{\partial x_1}\cdot\frac{\partial\phi_2}{\partial x_2}\cdot\frac{\partial\phi_3}{\partial x_3}\cdots\frac{\partial\phi_n}{\partial x_n}.$$

§ 188. *If* $y_1, \dots, y_n$ *be independent functions of* $x_1, \dots, x_n$ *then*

$$\frac{d(y_1, \dots, y_n)}{d(x_1, \dots, x_n)} \times \frac{d(x_1, \dots, x_m)}{d(y_1, \dots, y_m)} = \frac{d(y_{m+1}, \dots, y_n)}{d(x_{m+1}, \dots, x_n)}.$$

For the left-hand member

$$=\frac{d(y_1, \dots, y_n)}{d(x_1, \dots, x_n)} \times \frac{d(x_1, \dots, x_m, x_{m+1}, \dots, x_n)}{d(y_1, \dots, y_m, x_{m+1}, \dots, x_n)} \qquad (§ 183)$$

$$=\frac{d(y_1, \dots, y_m, y_{m+1}, \dots, y_n)}{d(y_1, \dots, y_m, x_{m+1}, \dots, x_n)} \qquad (§ 184)$$

$$=\frac{d(y_{m+1}, \dots, y_n)}{d(x_{m+1}, \dots, x_n)}. \qquad (§ 183)$$

§ 189. *If* $y_1, \dots, y_n$ *be functions of* $n+1$ *variables* $x_1, \dots, x_{n+1}$, *then*

$$\frac{\partial}{\partial x_1}\frac{d(y_1, \dots, y_n)}{d(x_2, \dots, x_n)} - \frac{\partial}{\partial x_2}\frac{d(y_1, \dots\dots\dots, y_n)}{d(x_1, x_3, \dots, x_{n+1})} + \dots + (-1)^n\frac{\partial}{\partial x_{n+1}}\frac{d(y_1, \dots, y_n)}{d(x_1, \dots, x_n)} = 0.$$

For each term on the left is expressible as the sum of n determinants (Ex. 30, p. 68), and the $n(n+1)$ determinants so obtained vanish in pairs, in virtue of the identity

$$\frac{\partial^2 y_m}{\partial x_r \partial x_s} = \frac{\partial^2 y_m}{\partial x_s \partial x_r}.$$

§ 190. *If* n *functions of* $x_1, x_2, \ldots, x_n$ *be not independent, their Jacobian with respect to* $x_1, x_2, \ldots, x_n$ *vanishes.*

Let the functions be $y_1, y_2, \ldots, y_n$, and the relation between them

$$\psi(y_1, y_2, \ldots, y_n) = 0.$$

Differentiating with respect to $x_1, x_2, \ldots, x_n$ in succession we have

$$\frac{\partial\psi}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1} + \frac{\partial\psi}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1} + \ldots + \frac{\partial\psi}{\partial y_n}\cdot\frac{\partial y_n}{\partial x_1} = 0$$

$$\frac{\partial\psi}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_2} + \frac{\partial\psi}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_2} + \ldots + \frac{\partial\psi}{\partial y_n}\cdot\frac{\partial y_n}{\partial x_2} = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial\psi}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_n} + \frac{\partial\psi}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_n} + \ldots + \frac{\partial\psi}{\partial y_n}\cdot\frac{\partial y_n}{\partial x_n} = 0$$

whence by elimination (pp. 75, 76) of $\frac{\partial\psi}{\partial y_1}, \frac{\partial\psi}{\partial y_2}, \ldots, \frac{\partial\psi}{\partial y_n}$ the desired result is obtained.

§ 191. *If the Jacobian of a set of functions vanishes the functions are not independent.*

Denoting the functions by $y_1, \ldots, y_n$ and the variables by $x_1, \ldots, x_n$ we have already seen that $y_1, \ldots, y_n$ are expressible in the forms $\phi_1(x_1, \ldots, x_n)$, $\phi_2(y_1, x_2, \ldots, x_n)$, $\phi_3(y_1, y_2, x_3, \ldots, x_n)$, $\ldots$, $\phi_n(y_1, \ldots, y_{n-1}, x_n)$ respectively, and that

$$\frac{d(y_1, \ldots, y_n)}{d(x_1, \ldots, x_n)} = \frac{\partial\phi_1}{\partial x_1}\cdot\frac{\partial\phi_2}{\partial x_2}\cdots\frac{\partial\phi_n}{\partial x_n}.$$

In the present case therefore one of the factors on the left, the m^{th} say, vanishes. Consequently ϕ_m does not involve x_m, that is, y_m is a function of $y_1, \ldots, y_{m-1}, x_{m+1}, \ldots, x_n$. But from this it follows that x_{m+1} is a function of $y_1, \ldots, y_m, x_{m+2}, \ldots, x_n$, hence y_{m+1}, which equals $\phi_{m+1}(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n)$, is a function of $y_1, \ldots, y_m, x_{m+2}, \ldots, x_n$. Similarly we may show that y_{m+2} is a function of $y_1, \ldots, y_{m+1}, x_{m+3}, \ldots, x_n$, and finally that y_n is a function of $y_1, \ldots, y_{n-1}$, which is what was to be proved.

§ 192. Closely connected with Jacobians is the determinant

$$\left| y_1, \frac{\partial y_2}{\partial x_1}, \frac{\partial y_3}{\partial x_2}, \ldots, \frac{\partial y_n}{\partial x_{n-1}} \right| \quad \text{or } I(y_1, y_2, \ldots, y_n)$$

whose first column consists of n functions of $n-1$ variables and which in every possible case has the element in its r^{th} row and s^{th} column equal to the differential coefficient of the r^{th} function with respect to the $(s-1)^{\text{th}}$ variable. Its most important property is that it is expressible by means of a Jacobian, viz. we have

$$I(y_1, y_2, \ldots, y_n) = y_1^{n-1} J\left(\frac{y_2}{y_1}, \frac{y_3}{y_1}, \ldots, \frac{y_n}{y_1}\right).$$

This is an immediate consequence of Ex. 20, p. 77, and the fact that

$$\frac{d}{dx}\left(\frac{y_r}{y_1}\right) = \left(y_1 \frac{dy_r}{dx} - y_r \frac{dy_1}{dx}\right) \div y_1^2.$$

§ 193. The Jacobian of the first differential coefficients of a function of n variables is called the HESSIAN of the function; in symbols,

$$H(u) = J\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right) = \left| \frac{\partial^2 u}{\partial x_1 \partial x_1}, \frac{\partial^2 u}{\partial x_2 \partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_n \partial x_n} \right|.$$

It is axi-symmetric in virtue of the identity

$$\frac{\partial^2 u}{\partial x_r \partial x_s} = \frac{\partial^2 u}{\partial x_s \partial x_r}.$$

§ 194. If there be n functions of one and the same variable x, the determinant which has in every case the element in its r^{th} row and s^{th} column the $(r-1)^{\text{th}}$ differential coefficient of the s^{th} function may be called the **Wronskian** of the functions with respect to x. Thus the Wronskian of y_1, y_2, y_3 with respect to x is

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \frac{dy_3}{dx} \\ \frac{d^2y_1}{dx^2} & \frac{d^2y_2}{dx^2} & \frac{d^2y_3}{dx^2} \end{vmatrix} \quad \text{or} \quad \left| y_1, \frac{dy_2}{dx_1}, \frac{d^2y_3}{dx^2} \right|.$$

It may be more shortly denoted by

$$W_x(y_1, y_2, y_3) \quad \text{or} \quad |y_{1(0)}\, y_{2(1)}\, y_{3(2)}|,$$

the enclosed suffixes referring to differentiations.

§ 195. *The differential coefficient of a Wronskian is got by differentiating each element of the last row.*

For, of the determinants, whose sum is in general the differential coefficient, all vanish (§ 27) except the last.

§ 196. *If* $y_1, \dots, y_n$ *be functions of* x, *and* x *be a function of* t, *then*

$$W_x(y_1, \dots, y_n) = \left(\frac{dt}{dx}\right)^{\frac{1}{2}n(n-1)} W_t(y_1, \dots, y_n).$$

Changing the independent variable in each element of the left-hand member we readily obtain the result on the right by the application of §§ 26, 32.

§ 197. *If having* n *functions we form three Wronskians, firstly that of all the functions except the last, secondly that of all except the second last, and thirdly that of all except the last two, then the quotient of the Wronskian of the first and second of these by the third is equal to the Wronskian of the* n *functions.*

Taking, for shortness, the case of four functions we have (§ 91 or pp. 140, 141)

$$\begin{vmatrix} u & v & w & y \\ u_{(1)} & v_{(1)} & w_{(1)} & y_{(1)} \\ u_{(2)} & v_{(2)} & w_{(2)} & y_{(2)} \\ u_{(3)} & v_{(3)} & w_{(3)} & y_{(3)} \end{vmatrix} = \begin{Vmatrix} |uv_{(1)}w_{(2)}| & |uv_{(1)}y_{(2)}| \\ |uv_{(1)}w_{(3)}| & |uv_{(1)}y_{(3)}| \end{Vmatrix} \div |uv_{(1)}|$$

$$= W\big(|uv_{(1)}w_{(2)}|, |uv_{(1)}y_{(2)}|\big) \div |uv_{(1)}| \,(\S 195)$$

$$= W\big(W(u,v,w), W(u,v,y)\big) \div W(u,v).$$

This is the most important case of the following widely general theorem in Compound Wronskians.

§ 198. *If* $y_1, \ldots, y_n$ *be functions of one and the same variable, then*

$$W(y_1, \ldots, y_m, \ldots, y_n)$$

$$= W\left\{W(y_1, \ldots, y_m, y_{m+1}), W(y_1, \ldots, y_m, y_{m+2}), \ldots, W(y_1, \ldots, y_m, y_n)\right\}$$

$$\div \left\{W(y_1, \ldots, y_m)\right\}^{n-m-1}.$$

Transforming $W(y_1, \ldots, y_n)$ by means of Ex. 20, p. 77, and multiplying columnwise the resulting compound determinant by the determinant of the $(n-1)^{\text{th}}$ order

$$\begin{vmatrix} y_1 & 0 & -y_{1(2)} & -2y_{1(3)} & -3y_{1(4)} & \cdots \\ 0 & y_1 & y_{1(1)} & 0 & -2y_{1(3)} & \cdots \\ 0 & 0 & y_1 & 2y_{1(1)} & 2y_{1(2)} & \cdots \\ 0 & 0 & 0 & y_1 & 3y_{1(1)} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

in which the coefficient of any element added to that of the element below equals that of the element to the right of the latter, and then, to neutralize this multiplication, dividing each element of the product by y_1, we obtain

$$W(y_1, \ldots, y_n) = W\{W(y_1 y_2), W(y_1 y_3), \ldots, W(y_1 y_n)\} \div y_1^{n-1}.$$

The case for which $m=2$ being thus established, we establish by means of it the case for $m=3$, and so on, exactly as in § 93.

§ 199. *If a set of functions of the same variable be connected by a linear relation with coefficients which are constant with respect to the variable, the Wronskian of the functions vanishes.*

Let the linear relation be

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0.$$

Differentiating $n-1$ times we have in all n equations from which by elimination of $c_1, \ldots, c_n$ the required result is obtained.

§ 200. *If the Wronskian of a set of functions vanishes, the functions are connected by a linear relation with coefficients which are constant with respect to the independent variable.*

Let the functions be $y_1, \ldots, y_n$. Then from § 198

$$W\{W(y_1 y_2), W(y_1 y_3), \ldots, W(y_1 y_n)\} = 0;$$

so that, if the theorem holds for the case of $n-1$ functions, we have

$$a_1 W(y_1 y_2) + a_2 W(y_1 y_3) + \ldots + a_{n-1} W(y_1 y_n) = 0,$$

or

$$y_1 \left\{ a_1 \frac{d}{dx}\left(\frac{y_2}{y_1}\right) + a_2 \frac{d}{dx}\left(\frac{y_3}{y_1}\right) + \ldots + a_{n-1} \frac{d}{dx}\left(\frac{y_n}{y_1}\right) \right\} = 0,$$

and therefore by division and integration

$$a_1\left(\frac{y_2}{y_1}\right)+a_2\left(\frac{y_3}{y_1}\right)+\ldots+a_{n+1}\left(\frac{y_n}{y_1}\right)+a_0=0$$

or
$$a_0y_1+a_1y_2+a_2y_3+\ldots+a_{n-1}y_n=0;$$

that is to say, the theorem holds for the case of n functions. But it evidently holds for the case of 2 functions, therefore it holds generally.

EXERCISES. SET XIX.

Establish the identities

1. $I(uu_1,\ uu_2,\ \ldots,\ uu_n) \quad = u^nI(u_1,\ u_2,\ \ldots,\ u_n)$;
2. $J(uu_1,\ uu_2,\ \ldots,\ uu_n) \quad = 2u^nJ(u_1,\ \ldots,\ u_n)-u^{n-1}I(u,\ u_1,\ \ldots,\ u_n)$;
3. $J(u_1,\ u_2,\ \ldots,\ u_n) \quad = \frac{1}{2}u^{-n}J(uu_1,\ \ldots,\ uu_n)+\frac{1}{2}u^nJ(u^{-1}u_1,\ \ldots,\ u^{-1}u_n)$;
4. $J(\Sigma x_1,\ \ldots,\ \Sigma x_1x_2\ldots x_n) = J\Big((x_1,\ \ldots,\ x_n)^1,\ \ldots,\ (x_1,\ \ldots,\ x_n)^n\Big)$
   $= \zeta^{\frac{1}{2}}(x_1,\ x_2,\ \ldots,\ x_n).$

5. **Show that if** $y_1=\phi_1(y_2,\ \ldots,\ y_n,\ x_1,\ \ldots,\ y_n)$, $y_2=\phi_2(y_3,\ \ldots,y_n,\ x_1,\ \ldots,\ x_n)$, ………, $y_n=\phi_n(x_1,\ \ldots,\ x_n)$ **then**

$$\frac{d(y_1,\ \ldots,\ y_n)}{d(x_1,\ \ldots,\ x_n)}=\frac{d(\phi_1,\ \ldots,\ \phi_n)}{d(x_1,\ \ldots,\ x_n)}.$$

6. **If** u **be a function of** $x_1,\ \ldots,\ x_n,\ \xi_1,\ \ldots,\ \xi_n$ **show that**

$$\frac{d\left(\frac{\partial u}{\partial x_1},\ \ldots,\ \frac{\partial u}{\partial x_n}\right)}{d(\xi_1,\ \ldots,\ \xi_n)}=\frac{d\left(\frac{\partial u}{\partial \xi_1},\ \ldots,\frac{\partial u}{\partial \xi_n}\right)}{d(x_1,\ \ldots,\ x_n)}.$$

7. **If** u **be a function of** $x_1,\ \ldots,\ x_n$ **and** $x_1,\ \ldots,\ x_n$ **be independent functions of** $\xi_1,\ \ldots,\ \xi_n$, **show that**

$$\frac{d\left(\frac{\partial u}{\partial x_1},\ \ldots,\ \frac{\partial u}{\partial x_n}\right)}{d(\xi_1,\ \ldots,\ \xi_n)}\cdot\frac{d\left(\frac{\partial u}{\partial \xi_1},\ \ldots,\ \frac{\partial u}{\partial \xi_n}\right)}{d(x_1,\ \ldots,\ x_n)}=\frac{d\left(\frac{\partial u}{\partial x_1},\ \ldots,\ \frac{\partial u}{\partial x_n}\right)}{d(x_1,\ \ldots,\ x_n)}\cdot\frac{d\left(\frac{\partial u}{\partial \xi_1},\ \ldots,\ \frac{\partial u}{\partial \xi_n}\right)}{d(\xi_1,\ \ldots,\ \xi_n)}.$$

Establish the identities

8. $W(yy_1,\ yy_2,\ yy_3,\ \ldots,yy_n)=y^nW(y_1,\ y_2,\ \ldots,\ y_n)$;
9. $W(1,\ 2y,\ 3y^2,\ \ldots,\ ny^{n-1})=n!(n-1)!\ldots 1\left(\frac{dy}{dx}\right)^{\frac{1}{2}n(n-1)}.$

CHAPTER IV.

HISTORICAL AND BIBLIOGRAPHICAL SUMMARY.

§ 201. In dealing with the solution of a set of simultaneous linear equations with literal coefficients any inquiring student might experience a desire to find the law of formation of the functions which appear as numerator and denominator in the resulting values. The first mathematician who when thus occupied had genius enough to obtain a glimpse of the possibility of a Theory of Determinants was LEIBNITZ. His idea, published in 1693, does not however appear to have been developed by him, and it soon sank into oblivion. In 1750 CRAMER, independently of Leibnitz, came somewhat nearer the full conception. He gave a rule (§ 19), long afterwards known by his name, for the formation of the said functions; but like his predecessor he failed to make the most of the discovery. A simpler rule (§ 15) than Cramer's was given by BEZOUT in 1764, when studying the allied subject of Elimination: in other respects no advance was then made. In 1771, however, new ground was broken by VANDERMONDE, who made the first step towards a *notation*. Taking

$$\begin{matrix} \alpha & \alpha & \alpha & \\ a' & b' & c' & \cdots \end{matrix}$$

for the coefficients of the first equation

$$\begin{matrix} \beta & \beta & \beta & \\ a' & b' & c' & \cdots \end{matrix}$$

for those of the second and so on, he then denoted the functions by

$$\frac{\alpha\,|\,\beta}{a\,|\,b},\qquad \frac{\alpha\,|\,\beta\,|\,\gamma}{a\,|\,b\,|\,c},\qquad \ldots\ldots$$

and wrote the law of development in the form

$$\frac{\alpha\,|\,\beta}{a\,|\,b}=\frac{\alpha}{a}\cdot\frac{\beta}{b}-\frac{\alpha}{b}\cdot\frac{\beta}{a},$$

$$\frac{\alpha\,|\,\beta\,|\,\gamma}{a\,|\,b\,|\,c}=\frac{\alpha}{a}\cdot\frac{\beta\,|\,\gamma}{b\,|\,c}+\frac{\alpha}{b}\cdot\frac{\beta\,|\,\gamma}{c\,|\,a}+\frac{\alpha}{c}\cdot\frac{\beta\,|\,\gamma}{a\,|\,b},$$

&c. Vandermonde also gave the theorem (§ 36)

$$\frac{\alpha\,|\,\beta\,|\,\gamma}{a\,|\,b\,|\,c}=-\frac{\alpha\,|\,\beta\,|\,\gamma}{b\,|\,a\,|\,c}=\ldots\ldots;$$

noted the result of making 'two letters of the same alphabet equal' (§ 27); and gave the development equation

$$\frac{\alpha\,|\,\beta\,|\,\gamma\,|\,\delta}{a\,|\,b\,|\,c\,|\,d}=\frac{\alpha\,|\,\beta}{a\,|\,b}\cdot\frac{\gamma\,|\,\delta}{c\,|\,d}-\ldots\ldots+\frac{\alpha\,|\,\beta}{c\,|\,d}\cdot\frac{\gamma\,|\,\delta}{a\,|\,b},$$

and other instances of the theorem stated in modern form in § 77. Nearly simultaneously with Vandermonde, **LAPLACE** made the very same important advances. He gave the same theorems, and indeed it is his name which one of them (§ 77) still bears: he introduced a make-shift notation, writing

$$(abc) \text{ for } ab'c''+a''bc'+a'b''c-a''b'c-a'bc''-ab''c',$$

and he made a beginning of a nomenclature, calling such functions as the said (abc) *resultants.* In 1773 some service to the subject was rendered by **LAGRANGE**, who incidentally gave in the ordinary non-determinant notation certain identities which are now easily recognizable as the special case of § 69 and of § 96 where $n=3$. In the same indirect way the subject occurs in 1779 in a second work by Bezout, where there are given many simple special instances of theorems, such as that of § 85. Passing over Rothe and other writers of the Hindenburg school we next come to Gauss (1801), whose connection with the subject was also quite similar to that of Lagrange. The terms '*Determinant*' and '*adjugate*' have their origin in Gauss's work. He defined

$$b^2 - ac,$$
$$ab^2 + a'b'^2 + a''b''^2 - aa'a'' - 2bb'b'',$$

as the *determinants* of

$$ax^2 + 2bxy + cy^2,$$
$$ax^2 + a'x'^2 + a''x''^2 + 2bx'x'' + 2b'xx'' + 2b''xx',$$

respectively, and the name thus given to a special form of the functions was afterwards adopted for the functions in general. Under the name of the "fonctions *Schin*" (ש) the subject was apparently familiar to Wronski in 1811: he did not however treat of it directly till 1815, and by that time he had been forestalled by the first great master of it, Cauchy.

§ 202. Cauchy's standpoint was entirely different from that of any of his predecessors. His memoir (1812) deals professedly with *Alternating Functions*, and at the end of the First Part, the subject of which is alternating functions in general, he intimates that he will now examine an important special class of these functions, instances of which have appeared in the solution of simultaneous linear equations, in the theory of elimination, and in the investigation of the properties of binary forms. He refers to Laplace, Vandermonde, Bezout and Gauss, and says he will adopt from the latter the name 'determinant' for functions of the special kind referred to. The Second Part upon which he then enters is nothing short of a methodically arranged treatise on determinants, extending to about 60 quarto pages. His definition is of course that which is founded on the theorem of § 118: his notation is

$$S(\pm a_{1\cdot1}\, a_{2\cdot2}\, a_{3\cdot3} \ldots a_{n\cdot n}) \quad \text{and} \quad (a_{1\cdot n}).$$

He arranges $a_{1\cdot1}$, $a_{1\cdot2}$, ... in a square; speaks of 'lignes horizontales' and 'colonnes verticales'; calls $a_{2\cdot3}$ and $a_{3\cdot2}$ 'conjugués'; applies the word 'principal' to $a_{1\cdot1}$, $a_{2\cdot2}$, ..., $a_{n\cdot n}$ and to their product; denotes the co-factor of $a_{\mu,\nu}$ by $b_{\nu,\mu}$, and forming the determinant (b_{1n}) calls it 'le système adjoint au système (a_{1n})'. The theorems of §§ 24, 36, 46, 52, 62 are established by him in order, and the example of pp. 73, 74 is given as an instance of the application of the theory. Further on we find the theorem of § 95, followed by a special case of the theorem of § 96, and towards the end a more important result still, the multiplication theorem (§ 67). He even enters on the subject of the compounds *(systèmes dérivés)* of (a_{1n}) and obtains the identity

$$\Delta_m \Delta_{n-m} = \Delta^{C_{n,m}}$$

which appears on page 211.

In the light of all this and bearing in mind the isolated character of the results obtained before his time, it is not to be wondered that Cauchy has been claimed as the real founder of the theory of determinants. His predecessors had left scattered here and there stones of varied mass and usefulness; Cauchy brought them together, laid the foundation, and made progress with the superstructure.

§ 203. Simultaneously with Cauchy, BINET obtained some of the results just mentioned, the most notable being the multiplication theorem. He calls the functions 'resultants', following Laplace instead of Gauss. In 1815 Wronski discussed the Schin functions at considerable length, as has been already stated. In 1821 Cauchy, returning to the subject (which indeed he did more than once, even as late as 1847) gave a short exposition of it in his Course of Analysis for the Polytechnic School; and from about this time forward determinants were never long lost sight of. Advantage was taken of them in analytical and geometrical investigations by JACOBI, REISS (1829, 1838), LEBESGUE (1837) and CATALAN (1839). For twenty years Jacobi's writings alone would have sufficed to keep the subject before the world. In memoirs on various matters (1827, 1833, 1835 &c.) we find him repeatedly using determinants, and at length in 1841 he made them the subject of a masterly monograph. After Cauchy's his is the next great name.

§ 204. Interest in determinants would now, doubtless, never have declined: but a sudden powerful impulse was given to the study of them by the researches which the English mathematicians began about 1840 into the theory of the linear transformation of quantics. In this theory the great instrument is determinants; and men who, like Cayley and Sylvester, worked with the instrument from day to day were sure to have new properties of it and new special forms of it brought before their notice. With Cayley, there came into use what we may call 'determinant brackets', viz. the now familiar pair of upright lines, and the determinant of a system of quantities was denoted by the said system itself. This facilitated the study of special forms: and, speaking generally, the work of the forty years from then till now has been work of this kind.

§ 205. The originator of the theory and application of Continuants was Sylvester. A number of the identities in Simple Continuants were however first given in non-determinant form by Euler.

§ 206. The first start in the theory of Alternants, as may be inferred from § 202 was made by Cauchy. Since his time Jacobi, Trudi, Nägelsbach and Garbieri have made the most important advances. The name 'Alternant' is adopted from Sylvester.

§ 207. Perhaps the first theorem regarding Axi-symmetric determinants occurs in a paper of Jacobi's: almost all that is known of them is due to Lebesgue, Sylvester and Hesse.

Persymmetric determinants were so called by Sylvester. They were afterwards studied by Hankel who gave the expression in non-determinant notation for a large number of special instances.

The first theorem regarding Circulants was given by Catalan: the fundamental theorem (§ 149) however is due to Spottiswoode. Additions to the theory have recently been made by Glaisher and Scott.

For the conception of Centro-symmetric determinants we are indebted to Zehfuss, who also gave the fundamental theorem regarding them.

§ 208. The author of the important theory of Skew Determinants and Pfaffians is Cayley. The latter functions first occur with their law of development in a paper by Jacobi dealing with the subject of Pfaff's problem; hence the name which Cayley gave to them.

§ 209. Compound Determinants occur in Cauchy, as has been already noted; the name however and the origination of the theory are due to Sylvester. Of subsequent investigators the most eminent are Reiss and Picquet: the admirable but neglected work of the former probably contains all that is as yet known on the subject.

§ 210. Jacobians were so called by Sylvester in well-deserved honour of Jacobi, who was the first to introduce them and who investigated their properties so thoroughly that little has been added to the theory since his time. The second definition of a Jacobian (§ 182) is due to Bertrand: the notation following upon this to Donkin.

Wronskians were first used by Wronski and appear in his well-known expansion-theorem. After being almost unheard of for about sixty years they have recently come into marked prominence through the researches of Christoffel and Frobenius, to whom in the main we are indebted for the discovery of their properties. Wronski's early connection with determinants has been so long unrecognised, and it is so convenient to have a short name for functions of which we repeatedly require to speak, that I think mathematicians will be glad to adopt the designation which I have ventured to propose.

§ 211. The first separately published treatise on Determinants was written by Spottiswoode, and appeared in 1851. After this came Brioschi's in 1854, Baltzer's (still a standard work) in 1857, Salmon's in 1859, Trudi's in 1862, Garbieri's in 1874, Günther's in 1875, Dostor's in 1877, Baraniecki's (the most extensive of all) in 1879, and Scott's in 1880. During the same time a large number of smaller works suited for schools have also appeared: good specimens are Mansion's in French, Bartl's in German, Mollame's in Italian, and Thomson's in English.

§ 212. The student who desires detailed information regarding the History of the subject should consult the memoirs of Mellberg and Studnička and the above-mentioned text-book of Günther.

§ 213. Perhaps the fullest possible materials for both a complete Theory and a complete History are to be found in the chronologically arranged "List of Writings on Determinants" (1693-1880) published by me in the Quarterly Journal of Mathematics for October, 1881.

RESULTS OF THE EXERCISES.

SET I.

1. $cgm + efl + dhk - egk - dfm - chl.$ 2. $swx + rvz + tuy - twz - svy - rux.$

3. $a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$

4. $aek - cdh + bfg - ceg + bdk - afh.$

5. $4abc + 9b^2d - 16acd + 2bc^2 - 24ab^2 - 12ad^2.$ 6. $aeh - bfg + ceg.$

7. $x^2y + xy^2.$ 8. $2(x^4 + y^4 + z^4 + x^2yz + xy^2z + xyz^2).$ 9. $a^3 + b^3 + c^3 - 3abc.$

10. 0. 11. $\frac{1}{2160}$. 12. −89. 13. 12. 14. 66. 15. 47.

16. $\begin{vmatrix} b & c & d \\ e & f & g \\ h & i & j \end{vmatrix}.$ 17. $\begin{vmatrix} m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \\ r_1 & r_2 & r_3 \end{vmatrix}.$

18. 3. 19. $a, b.$

20. $2(a^2 + ab + b^2) \div 3(a + b).$

21. $8abc.$

SET II.

1. Their sum is zero. 2. They are equal in magnitude, but the second differs in sign. 3. See § 34. 6. See § 27.

8. $z_1\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} - z_2\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + z_3\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \quad -x_2\begin{vmatrix} y_1 & z_1 \\ y_3 & z_3 \end{vmatrix} + y_2\begin{vmatrix} x_1 & z_1 \\ x_3 & z_3 \end{vmatrix} - z_2\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}.$

14. $\begin{vmatrix} a & b & c \\ x & y & z \\ m & n & p \end{vmatrix}.$ 15. $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}.$ 16. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$ 17. $\begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}.$

18. $\begin{vmatrix} x & a & b \\ a & x & c \\ b & c & x \end{vmatrix}.$ 19. $\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}.$ 20. $\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix}.$

SET III.

1. 3, 4. 2. 4, 1. 3. 3, 3. 4. 5, 1. 5. $-\dfrac{a^2 + ab + b^2}{a + b}, \ -\dfrac{ab}{a + b}.$

6. 1, 2. 7. 1, 2, 3. 8. 2, 3, 4. 9. 5, 1, 3. (For '$-3x$' read '$-3z$.')

10. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}.$ 11. 6, 2, −3. 12. $\frac{83}{49}, \frac{332}{161}, \frac{166}{237}.$

SET IV.

1. $(-dfk + dgj + chj + cfl - bgl)m$.
2. $(-glm + elo - hio + gip - ekp)b$.
3. $(-dfm + afp - bep + den - ahn)k$.
4. $(-bio + cin - akn + ajo - cjm)h$.
5. $(dek - dig + agl - ahk + chi - cel)n$.
6. $(bel - bhi + dif - dej + ahj - afl)o$.
7. $(afk - agj + bgi - bek + cej - cfi)p$.
8. $(y_4z_3v_2 - y_4z_2v_3 + z_4y_2v_3 - z_4y_3v_2 + v_4y_3z_2 - v_4y_2z_3)x_5w_1$.
9. $(x_1z_4v_5 - x_1z_5v_4 + x_4z_5v_1 - x_4z_1v_5 + x_5z_1v_4 - x_5z_4v_1)y_2w_3$.
10. $(y_2z_3w_5 - y_2z_5w_3 + y_3z_5w_2 - y_3z_2w_5 + y_5z_2w_3 - y_5z_3w_2)x_4v_1$.

SET V.

1. 9, 15, 8, 14, 16, 17.
2. −, −, +, −.
3. $afkp - aflo + agln - agjp + ahjo - ahkn + bhkm - bglm + belo - bhio + bgip - bekp + cejp - cfip + chin - celn + cflm - chjm + dgjm - dfkm + dekn - dign + difo - dejo$.
4. +, −, +, +.

SET VI.

1. $a_1b_1c_1d_1$.
2. 0.
3. $b_2c_3d_4e_5$.
4. $x^5 + y^5$.
5. x^5.
6. $(-x_0z_1 + x_1z_0)y_2w_3$.
7. $-b_2c_3d_4e_1f_0$.
8. $\frac{1}{2}n(n-1)$.
9. $a_rb_pc_md_ne_q$.
10. +, if n be of the forms $4n$, $4n+1$; −, if not. Or, $(-1)^{\frac{1}{2}n(n-1)}$.
11. $(a_1c_6d_3 - a_3c_6d_1 + a_3c_1d_6 - a_6c_1d_3 + a_6c_3d_1 - a_1c_3d_6)b_5$.
12. Number unknown: it is however an *odd* number.
13. $(n-1)!$.

SET VII.

1. −398.
2. 911.
3. 30.
4. 6904.
5. 2106.
6. 2660.
7. The determinant is multiplied by $(-1)^n$.
8. 0.
10. 0, 0.
10. The last element was meant to be 0; and this change being made the result is

$$\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$

15. $-|a_1c_2d_3| \div |b_1c_2d_3|$.
17. 0, 0.
18. $|a_2b_3c_4|x^2 + |a_1b_3c_4|x + |a_0b_3c_4|$.
21. $(a+b+c)(a-b+c)(a-b-c)(a+b-c)$.
24. $(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)$.
25. $(x+3)(x-1)^3$.
26. $-a(a-b)(b-c)(c-d)$.
27. $(dz-cw)(dy-bw)(dx-aw)(cy-bz)(cx-az)(bx-ay)$.
29. $\begin{vmatrix} a_1+a_2-a_3 & a_4 & a_5 \\ b_1+b_2-b_3 & b_4 & b_5 \\ c_1+c_2-c_3 & c_4 & c_5 \end{vmatrix}.$
30. $\begin{vmatrix} a_0+a_1 & a_2-a_3 & a_5 \\ b_0+b_1 & b_2-b_3 & b_5 \\ c_0+c_1 & c_2-c_3 & c_5 \end{vmatrix}.$

SET VIII.

6. $|d_3e_4f_0b_1c_2|$. 7. $|e_1f_2b_3c_4d_0|$. 8. $|d_1e_2f_3b_4c_0|$. 9. $|e_2f_3b_4c_0d_1|$.

13. $-\begin{vmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{vmatrix}, \quad \begin{vmatrix} b & a & d & c \\ c & b & a & d \\ d & c & b & a \\ a & d & c & b \end{vmatrix}, \quad -\begin{vmatrix} c & b & a & d \\ d & c & b & a \\ a & d & c & b \\ b & a & d & c \end{vmatrix}, \quad \begin{vmatrix} d & c & a & b \\ a & d & c & b \\ b & a & d & c \\ c & b & a & d \end{vmatrix}.$

SET IX.

1. -101. 2. -336. 3. 377. 4. -240. 5. -2660.
6. -118424. 7. 172. 8. 22692. 9. 14940. 10. 12228.
11. x^5. 12. $a_1b_2c_3d_4e_5$. 13. $abcd+ab+ad+cd+1$. 14. $-3abcd$.
15. $a^2\alpha^2+b^2\beta^2+c^2\gamma^2-2ab\alpha\beta-2bc\beta\gamma-2ca\gamma\alpha$. 16. $(x-a)^3$. 17. $(x-y)(x^2+y^2)$.
18. $a^2+b^2+c^2-2ab-2bc-2ca$. 19. $8abcd$. 20. $abcd(1+a^{-1}+b^{-1}+c^{-1}+d^{-1})$.
21. $acde+bcde+bcd^2+b^2de+bc^2e$. 22. $a_n+a_{n-1}x+a_{n-2}x+\ldots+a_0x^n$.
23. $(-1)^n b_2b_3\ldots b_n(a_1b_1+a_2b_2+\ldots+a_{n+1}b_{n+1})$. 28. $(x-a_1)(x-a_2)(x-a_3)(x-a_4)$.

30. $\begin{vmatrix} \partial a_1 & a_2 & a_3 & a_4 \\ \partial b_1 & b_2 & b_3 & b_4 \\ \partial c_1 & c_2 & c_3 & c_4 \\ \partial d_1 & d_2 & d_3 & d_4 \end{vmatrix} + \begin{vmatrix} a_1 & \partial a_2 & a_3 & a_4 \\ b_1 & \partial b_2 & b_3 & b_4 \\ c_1 & \partial c_2 & c_3 & c_4 \\ d_1 & \partial d_2 & d_3 & d_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & \partial a_3 & a_4 \\ b_1 & b_2 & \partial b_3 & b_4 \\ c_1 & c_2 & \partial c_3 & c_4 \\ d_1 & d_2 & \partial d_3 & d_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 & \partial a_4 \\ b_1 & b_2 & b_3 & \partial b_4 \\ c_1 & c_2 & c_3 & \partial c_4 \\ d_1 & d_2 & d_3 & \partial d_4 \end{vmatrix}.$

SET X.

1. -202. 2. -238. 3. 10173. 4. -626.
5. $x=1,\ y=3,\ z=-1,\ w=3$. 6. $x=2,\ y=4,\ z=-1,\ w=-3$.
7. $x=a+d-e,\quad y=b+e-a,\quad z=c+a-b,\quad v=d+b-c,\quad w=e+c-d$.
8. $x=(5\tfrac{1}{2}S-16e+8a-4b+2c-d)\div 33,\ y=(5\tfrac{1}{2}S-16a+8b-4c+2d-e)\div 33,\ \ldots,$
if $S=a+b+c+d+e$. 9. $x=\dfrac{(d-e)(c-e)(b-e)}{(d-a)(c-a)(b-a)},\ y=\dfrac{(d-e)(c-e)(a-e)}{(d-b)(c-b)(a-b)},\ \ldots$.
10. $x=\dfrac{abc}{(c-1)(b-1)(a-1)},\ y=\dfrac{be}{(c-a)(b-a)(1-a)},\ z=\dfrac{ca}{(c-b)(a-b)(1-b)},\ w=\dfrac{ab}{(1-c)(a-c)(b-c)}$.
11. $x=a+b-2c+3d-2e,\qquad y=b+c-2d+3e-2a,\ \ldots\ldots$

12. $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ a & c & b & d \\ c & a & d & b \end{vmatrix}=0.$ 13. $\begin{vmatrix} 0 & a_1 & b_1 & c_1 & d_1 \\ a_1 & b_1 & c_1 & d_1 & 0 \\ 0 & 0 & b_2 & c_2 & d_2 \\ 0 & b_2 & c_2 & d_2 & 0 \\ b_2 & c_2 & d_2 & 0 & 0 \end{vmatrix}=0.$ 14. $\begin{vmatrix} 0 & a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 & 0 \\ 0 & a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 & 0 \end{vmatrix}=0.$

15. $$\begin{vmatrix} 0 & a_1 & b_1 & c_1 & d_1 & e_1 \\ a_1 & b_1 & c_1 & d_1 & e_1 & 0 \\ 0 & 0 & 0 & c_2 & d_2 & e_2 \\ 0 & 0 & c_2 & d_2 & e_2 & 0 \\ 0 & c_2 & d_2 & e_2 & 0 & 0 \\ c_2 & d_2 & e_2 & 0 & 0 & 0 \end{vmatrix} = 0.$$

19. $$\begin{vmatrix} 0 & a_1 & 0 & b_1 & c_1 \\ a_1 & 0 & b_1 & c_1 & 0 \\ 0 & 0 & d_1 & e_1 & f_1 \\ 0 & d_1 & e_1 & f_1 & 0 \\ d_1 & e_1 & f_1 & 0 & 0 \end{vmatrix} = 0.$$

SET XI.

1. $|a_0c_1d_2e_4f_5|$. 2. $|a_0b_1d_3f_5|$. 3. $|a_1b_2d_3e_4|$. 4. $|a_1d_2e_3f_4|$.

5. $|a_0b_1c_5|$. 6. $|a_0e_2f_5|$. 7. $|a_1c_3d_5|$. 8. $|a_1b_3f_5|$. 9. $|a_{01}a_{12}a_{44}a_{56}|$, $|a_{01}a_{23}a_{44}|$.

10. $|a_{11}a_{33}a_{44}a_{66}|$, $-|a_{01}a_{24}a_{56}|$. 11. $-|a_{31}a_{54}a_{65}|$, $-|a_{01}a_{14}a_{35}a_{56}|$.

12. $|a_{11}a_{22}\ldots a_{m,m}\ldots a_{rr}\ldots a_{nn}| = -|a_{11}a_{22}\ldots a_{rm}\ldots a_{mr}\ldots a_{nn}|$. 13. 2.

14. $(bg-cf+de)^2$. 15. $x^{16}+x^{12}+x^{10}+2x^8+x^6+x^4+1$.

16. $(f^2-bc)x^2+(g^2-ac)y^2+(h^2-ab)z^2+2(hc-gf)xy+2(af-gh)yz+2(bg-hf)zx$.

17. $4(x^2+y^2+z^2+w^2-2xy-2xz-2xw-2yz-2yw-2zw)$.

19. $-(ax^2+by^2+cz^2+2hxy+2fyz+2gzx)$. 20. $\left\{\tfrac{1}{2}n(n-1)\right\}^2$.

22. $(n-k)!\left\{\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}-\ldots+(-1)^{n-k}\frac{1}{(n-k)!}\right\}$.

23. $n-1$, the number of positive terms being greater or less according as n is odd or even.

SET XII.

1. $$\begin{vmatrix} a_1x+a_2y+a_3 & b_1x+b_2y+b_3 & c_1x+c_2y+c_3 \\ a_1x_1+a_2y_1+a_3 & b_1x_1+b_2y_1+b_3 & c_1x_1+c_2y_1+c_3 \\ a_1x_2+a_2y_2+a_3 & b_1x_2+b_2y_2+b_3 & c_1x_2+c_2y_2+c_3 \end{vmatrix}, \text{ etc.}$$

2. $$\begin{vmatrix} ab & ac & ae+bf \\ bd & c^2+d^2 & ce \\ ae+bf & df & ef \end{vmatrix}, \text{ etc.}$$

3. $$\begin{vmatrix} a^2 & a^2-ab+b^2 & a^2-ac+c^2 \\ a^2-ab+b^2 & b^2 & b^2-bc+c^2 \\ a^2-ac+c^2 & b^2-ab+c^2 & c^2 \end{vmatrix}, \text{ etc.}$$

4. $$\begin{vmatrix} (a+b)^2 & a^2 & b^2 \\ c^2 & (b+c)^2 & b^2 \\ c^2 & a^2 & (c+a)^2 \end{vmatrix}, \text{ etc.}$$

5. $(xyz-yz^2-xy^2-x^2z)^2+(xz^2+x^2y+y^2z+xyz)^2$.

6. $a(b-a)(c-b)(d-c)$.

7. $$\begin{vmatrix} 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \end{vmatrix}.$$

8. $$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}.$$

9. $$\begin{vmatrix} x & y & 1 \\ y & 1 & x \\ 1 & x & y \end{vmatrix}.$$

14. $-\lambda^4+\lambda^4(a^2+b^2+c^2+2f^2+2g^2+2h^2)$

$$-\lambda^2\left\{\begin{vmatrix} h & b & f \\ g & f & c\end{vmatrix}^2+\begin{vmatrix} a & h & g \\ g & f & c\end{vmatrix}^2+\begin{vmatrix} a & h & g \\ h & b & f\end{vmatrix}^2\right\}+\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c\end{vmatrix}^2.$$

16. $\begin{vmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & f\end{vmatrix}^2.$ 18. $|a_1b_2c_3|\ |x_1y_2|.$ 18. $\begin{vmatrix} a & b & c \\ 0 & d & 0 \\ f & 0 & g\end{vmatrix}\times\begin{vmatrix} x_1 & 0 & z_1 \\ x_2 & y_2 & z_2 \\ 0 & y_3 & z_3\end{vmatrix}.$

SET XIII.

2. $z_1|y_1z_3|\ |x_1y_2z_3|.$ 3. $-z_1|x_1y_2z_3|\ |y_1z_2|.$ 4. $|a_1b_4|\ |a_2b_5|\ |a_3b_6|.$

8. $(p_2-p_1)(a_2-a_1)x_2x_1+(p_3-p_1)(a_3-a_1)x_3x_1+(p_4-p_1)(a_4-a_1)x_3x_2$
$+(p_3-p_2)(a_3-a_2)x_4x_1+(p_4-p_2)(a_4-a_2)x_4x_2+(p_4-p_3)(a_4-a_3)x_4x_3.$

10. $C_{n,a}\times C_{\beta+\gamma,\beta}.$ 26. $\begin{vmatrix} a & b \\ c & d\end{vmatrix}^4.$

SET XV.

1. $\begin{vmatrix} 1 & x_2+x_3+x_4 & x_2x_3+x_3x_4+x_4x_2 & x_2x_3x_4 \\ 1 & x_3+x_4+x_1 & x_3x_4+x_4x_1+x_1x_3 & x_3x_4x_1 \\ 1 & x_4+x_1+x_2 & x_4x_1+x_1x_2+x_2x_4 & x_4x_1x_2 \\ 1 & x_1+x_2+x_3 & x_1x_2+x_2x_3+x_3x_1 & x_1x_2x_3\end{vmatrix}.$

2. $\Sigma a^4bc+\Sigma a^3b^2c+\Sigma a^2b^2c^2.$
3. $\Sigma a^3b^2c+2\Sigma a^2b^2c^2.$
4. $\Sigma a^3bc+2\Sigma a^2b^2c+\Sigma a^3b^2.$

5. $\Sigma a^3b^3+\Sigma a^3b^2c+\Sigma a^2b^2c^2.$ 6. $\Sigma a^4b+\Sigma a^3b^2+2\Sigma a^3bc+2\Sigma a^2b^2c.$

7. $\Sigma a^3b^3+\Sigma a^3b^2c+\Sigma a^3bcd+\Sigma a^2b^2c^2+\Sigma a^2b^2cd.$

8. $|a^1\,b^m\,c^n\,d^p|+|a^0\,b^{m+1}\,c^n\,d^p|+|a^0\,b^m\,c^{n+1}\,d^p|+|a^0\,b^m\,c^n\,d^{p+1}|.$ 16. $\Sigma(a-b)^2.$

18. $(a+b+c+d)^3.$ 19. 4. 24. $\zeta^{\frac{1}{2}}(ab\ldots l)\times A_1A_2\ldots A_{n-1}.$ 25. $\{2!3!\ldots(n-1)!\}^{-1}.$

26. $m=0,\ n=1,\ p=2,\ q=3.$ 27. $C_{n-1,1}\times C_{n-1,2}\times\ldots\times C_{n-1,n-1}.$

29. $m+n+p+q-5.$ 33. $(-1)^{\frac{1}{2}n(n+1)}\{u_1u_2\ldots u_n\}^{-1}$, where $u_r=(x_r-y_1)(x_r-y_2)\ldots(x_r-y_n).$

34. $2^{2n^2}.$ 36. $(-1)^{\frac{1}{2}n(n+1)}\{v_1v_2\ldots v_n\}^{-1}$, where $v_r=(x_r-y_1)(x_r-y_2)\ldots(x_r-y_{n+1}).$

SET XVI.

1. Centro-symmetric. 5. $(a+h+d+e+b+g+c+f)(a+h+d+e-b-g-c-f$
$(a+h-d-e+b+g-c-f)(a+h-d-e-b-g+c+f)(a-h+d-e+b-g+c-f)$
$(a-h+d-e-b+g-c+f)(a-h-d+e+b-g-c+f)(a-h-d+e-b+g+c-f).$

9. $K(a+b^{-ab},a+b^{-ab},a+b^{-ab},a+b).$ 14. $a_{uv}=a_{u\pm t,v\mp t}.$

15. $\dfrac{(a-b)(x-y+c-d)^3(d-c)(x-y-a+b)^3}{a-b+c-d}\times\dfrac{(a+b)(x+y-c-d)^3-(c+d)(x+y-a-b)^3}{a+b-c-d}.$

24. $n\times$ complementary minor of a_1.

25. The next case is

$$\begin{vmatrix} 2a_1a_2-S & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 & a_1d_2+a_2d_1 \\ a_1b_2+a_2b_1 & 2b_1b_2-S & b_1c_2+b_2c_1 & b_1d_2+b_2d_1 \\ a_1c_2+a_2c_1 & b_1c_2+b_2c_1 & 2c_1c_2-S & c_1d_2+c_2d_1 \\ a_1d_2+a_2d_1 & b_1d_2+b_2d_1 & c_1d_2+c_2d_1 & 2d_1d_2-S \end{vmatrix} = -S^2(a_1^2+b_1^2+c_1^2+d_1^2)(a_2^2+b_2^2+c_2^2+d^2)$$

where $S = a_1a_2+b_1b_2+c_1c_2+d_1d_2$.

27. $(n-2p)(-2)^{n-1}$. 29. 0.

36. p or 0, according as n is or is not prime to p.

SET XVII.

3. $1.3.5\ldots(2n-1)$.

8. If the determinant be $D(a_{1n})^{a_{rr}=0}_{a_{rs}=-a_{sr}}$, we have $\dfrac{\partial D}{\partial a_{rs}} = 2D^{(r}_{(s}$ or 0 according as n is even or odd.

9. $\left|\begin{array}{ccc} a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & & c_4 \end{array}\right| + \left|\begin{array}{ccc} a_2 & a_3 & a_4 \\ & b_6 & b_5 \\ & & c_5 \end{array}\right| + \left|\begin{array}{ccc} a_7 & a_6 & a_5 \\ & b_3 & b_4 \\ & & c_4 \end{array}\right| + \left|\begin{array}{ccc} a_7 & a_6 & a_5 \\ & b_6 & b_5 \\ & & c_5 \end{array}\right|$.

14. The ω inside the second Pfaffian would require to be deleted, the ω outside changed to ω^2.

15. The next case is

$$\left|\begin{array}{cccc} (x_6-x_5)^5 & (x_6-x_4)^5 & \ldots & (x_6-x_1)^5 \\ & (x_5-x_4)^5 & \ldots & (x_5-x_1)^5 \\ & & \ldots\ldots & \ldots\ldots \\ & & & (x_2-x_1)^5 \end{array}\right| = -C_{5,1} \times C_{5,2} \times \zeta^{\frac{1}{2}}(x_1 \ldots x_6).$$

20. $(n_3a^2-n_2b^2+n_1c^2-n_1n_2n_3x^2)^2$.

SET XVIII.

1. 4th. 2. 4th row and 9th column. 3. $\begin{vmatrix} 1\,2\,3 \\ 6\,8\,9 \end{vmatrix}$. 7. $\begin{vmatrix} 1\,2\,3 \\ 1\,2\,3 \end{vmatrix}\begin{vmatrix} 1\,2\,3 \\ 4\,5\,6 \end{vmatrix}^2$.

9. $-|\alpha_1\beta_2\gamma_3||x_1y_2z_3|\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}^2$. 11. $\begin{vmatrix} 1\,2\,3\,4 \\ 1\,2\,3\,7 \end{vmatrix}\begin{vmatrix} 1\,2\,3\,4 \\ 4\,5\,6\,7 \end{vmatrix}$.

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